Math Bowl 2004 Solutions to Written Test

1. If $f(x) = ax^2 + bx + c$, $f(1) = 3$, and $f(-1) = 3$; then $a + c$ equals

$f(1) = a + b + c = 3$ **(1)**
$f(-1) = a - b + c = 3$ **(2)**
(**1**) + (**2**)
$2(a + c) = 6$
$a + c = 3$

Answer: C

2. The ratio of x to y is equal to one over their sum. Express y in terms of x.

$$\frac{x}{y} = \frac{1}{x + y}$$
$$x(x + y) = y$$
$$x^2 + xy = y$$
$$x^2 = y(1 - x)$$
$$y = \frac{x^2}{1 - x}$$

Answer: A

3. The area of a triangle is 30. If its base is 4 more than its height, what is the length of its height?

Let h be the height of the triangle.
$$30 = \frac{1}{2}h(h + 4)$$
$h^2 + 4h - 60 = 0$ $(h - 6)(h + 10) = 0$
$h = 6$

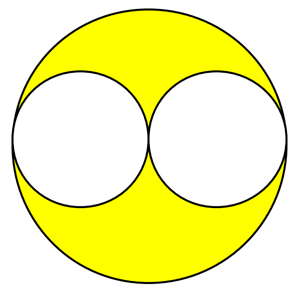
Answer: D

4. (Tie Break No. 1) If $\log_x 2 = a$, $\log_x 3 = b$, and $\log_x 5 = c$, find $\log_x \frac{24}{25}$ in terms of *a*, *b*, and *c*.

$$\log_x \frac{24}{25} = \log_x \frac{2^3 \cdot (3)}{5^2} = 3\log_x 2 + \log_3 3 - 2\log_x 5 = 3a + b - 2c$$

Answer: B

5. Each small circle has diameter $d = 6m$. The small circles are tangent to a large circle from inside; and tangent to each other at the center of the large circle. Find the area of the shaded region



The radius of the large circle is 6

The area of the large circle $A_1 = 6^2 \cdot p = 36p$

The radius of the small circle is 3

The area of the small circle $A_2 = 3^2 \cdot p = 9p$

The area of the shaded region

$A = A_1 - 2A_2 = 36p - 2 \cdot 9p = 18p\ m^2$

Answer: B

6. Solve the simultaneous equations $2x - 5y = 4$, and $3x + 2y = 25$; then use your answers to evaluate the expression $x^3 + 2y^2$.

$2x - 5y = 4$ **(1)**

$3x + 2y = 25$ **(2)**

$(\mathbf{1}) \times (-3) + (\mathbf{2}) \times 2$

$19y = 38$

$y = 2$ and $x = 7$

$x^3 + 2y^2 = 7^3 + 2 \cdot 2^2 = 351$

Answer: B

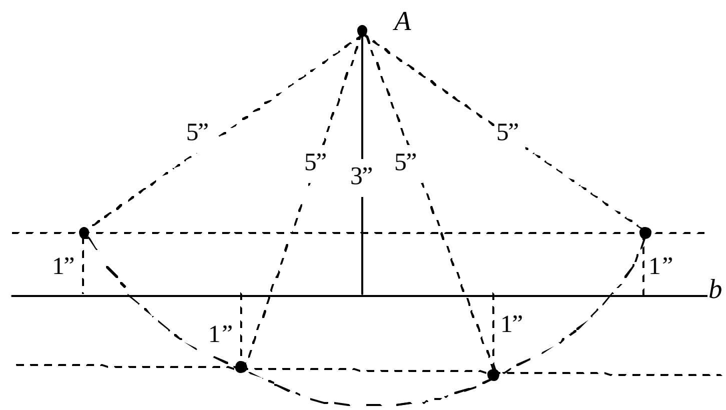
7. John Q. Ferris goes to ride the Ferris Wheel at the state fair. If the distance from the seat to the center of the wheel is 50 feet, and the wheel goes around 3 times every minute, find the speed in feet/sec of someone riding the wheel.

Speed $s = \dfrac{3 \cdot 2 \cdot 50p}{60} = 5p \;\; ft/\sec$

Answer: B

8. Point *A* is 3 inches from line *b* as shown in the diagram. In the plane that contains point *A* and line *b*, what is the total number of points which are 5 inches from *A* and also 1 inch from *b*?

As shown on the figure, there are four points satisfying the requirement



Answer: E

9. Bob and Ted are on a bike ride. Ted runs into a tree, ruining his bike. They are 16 *km* from home. They decide that Ted will start on foot and Bob will ride his bike. After awhile Bob will leave his bike beside the road and continue on foot so that Ted can ride the bike home when he gets to where Bob left the bike. Bob walks $4km/hr$ and bikes $10km/hr$ while Ted walks $5km/hr$ and bikes $12km/hr$. For what length of time should Ted ride the bike if they are both to arrive home at the same time?

Let *x* be the length of time for Ted to ride and *y* the length of time for him to walk.

$12x + 5y = 16 \qquad \textbf{(1)}$

$\dfrac{5y}{10} + \dfrac{12x}{4} = x + y$ or $4x - y = 0 \qquad \textbf{(2)}$

$(\mathbf{1}) + (\mathbf{2}) \times 5$

$32x = 16$

$x = \dfrac{1}{2} hr$

Answer: A

10. Let $\{a_1, a_2, a_3\}$ be a finite geometric sequence and $1 < a_1 < a_2 < a_3$. Then $\{\log_{a_1} 5, \log_{a_2} 5, \log_{a_3} 5\}$ has the property:

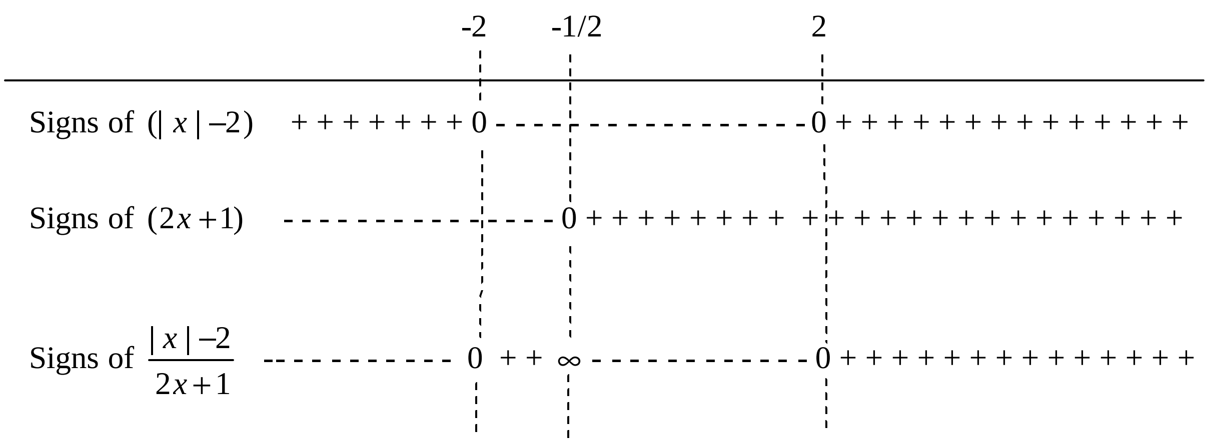
Let $a_1 = a$, $a_2 = ar$, and $a_3 = ar^2$

$\log_{a_1} 5 = \dfrac{\ln 5}{\ln a}$, $\log_{a_2} 5 = \dfrac{\ln 5}{\ln ar} = \dfrac{\ln 5}{\ln a + \ln r}$, and $\log_{a_3} 5 = \dfrac{\ln 5}{\ln ar^2} = \dfrac{\ln 5}{\ln a + 2\ln r}$

The reciprocals of $\log_{a_1} 5, \log_{a_2} 5,$ and $\log_{a_3} 5$ form an arithmetic sequence with common difference $\dfrac{\ln r}{\ln 5}$.

Answer: C

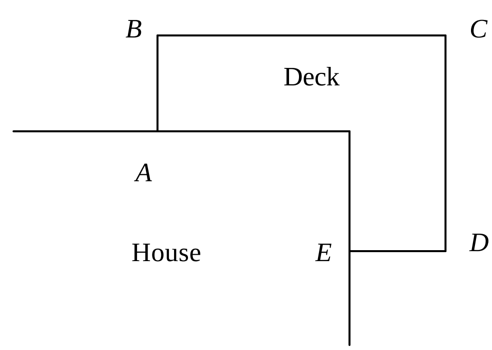
11. (Tie Break No.2) The domain of the function $f(x) = \log_a \dfrac{|x| - 2}{2x+1}$ is



The Domain of $f(x)$ is $(-2, -1/2) \cup (2, \infty)$

Answer: A

12. The Simpson's want to build a deck with a railing around a corner of their house. The railing will measure 30 meters, $AB = DE$, and $BC = CD$. What should AB measure to maximize the area of the deck?



Let $x = AB = DE$ and $y = BC = CD$

$2x + 2y = 30$ **(1)**

Area of the deck $A = y^2 - (y - x)^2$ **(2)**

Solve **(1)** for y and substitute the result into **(2)**.

$A = (15 - x)^2 - (15 - 2x)^2$

$= -3x^2 + 30x = -3(x-5)^2 + 75$

When $x = 5m$, A has the maximum value $75m^2$

Answer: A

13. Simplify $\dfrac{\sin 2x}{\sin^3 x + \sin x \cos^2 x}$

$$\frac{\sin 2x}{\sin^3 x+\sin x\cos^2 x}=\frac{2\sin x\cos x}{\sin x(\sin^2 x+\cos^2 x)}=\frac{2\sin x\cos x}{\sin x\cdot 1}=2\cos x$$

Answer: D

14. Find $x+y+z$ given that $\dfrac{x}{3-x}=\dfrac{y}{5-y}=\dfrac{z}{16-z}=2$

Solve $\dfrac{x}{3-x}=2$ for x $\quad x=2(3-x) \quad x=2$

Solve $\dfrac{y}{5-y}=2$ for y $\quad y=2(5-y) \quad y=\dfrac{10}{3}$

Solve $\dfrac{z}{16-z}=2$ for z $\quad z=2(16-z) \quad z=\dfrac{32}{3}$

$$x+y+z=2+\frac{10}{3}+\frac{32}{3}=16$$

Answer: B

15. If the $\tan x=\dfrac{2ab}{a^2-b^2}$ where $a>b>0$ and $0^\circ<x<90^\circ$, then $\sin x=?$

$$\sin^2 x=\frac{1}{\csc^2 x}=\frac{1}{\cot^2 x+1}=\frac{1}{\dfrac{1}{\tan^2 x}+1}=\frac{\tan^2 x}{\tan^2 x+1}$$

$$\sin^2 x=\frac{\left(\dfrac{2ab}{a^2-b^2}\right)^2}{\left(\dfrac{2ab}{a^2-b^2}\right)^2+1}=\frac{(2ab)^2}{(2ab)^2+(a^2-b^2)^2}=\frac{(2ab)^2}{4a^2b^2+a^4-2a^2b^2+b^4}=\left(\frac{2ab}{a^2+b^2}\right)^2$$

$$\sin x=\frac{2ab}{a^2+b^2}$$

Answer: E

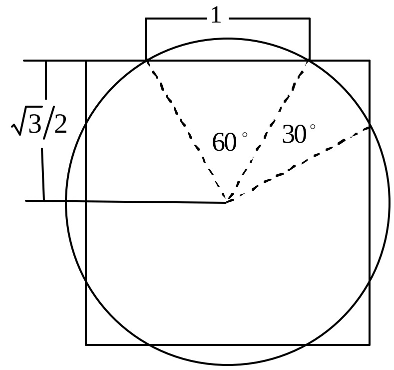
16. A circle with radius 1 and a square with side $\sqrt{3}$ have the same center. Find the area of the region overlapped by these two figures.

The region can be divided into four congruent triangles and four congruent sectors.

Area of the sector $A_1 = \frac{1}{2} \cdot 1^2 \cdot \frac{p}{6} = \frac{p}{12}$

Area of the triangle $A_2 = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{4}$

Area of the region $A = 4(A_1 + A_2) = \frac{p}{3} + \sqrt{3}$



Answer: B

17. It takes one day to fill the vat
With this large pipe, two days with that;
The third pipe needs but one day more;
The fourth pipe fills the vat in four.
If all four pipes together run,
How long before the task is done?

$$\frac{1}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = \frac{1}{\frac{25}{12}} = \frac{12}{25} \ day$$

Answer: B

18. R varies as the square of z and inversely as the cube of T. If z is tripled and T is doubled, the value of R is

$$R = k\frac{z^2}{T^3}$$

$$k\frac{(3z)^2}{(2T)^3} = \frac{9}{8}\left(k\frac{z^2}{T^3}\right)$$

Answer: B

19. Let $A = \begin{bmatrix} 7 & -5 & -8 \\ x & -2 & -3 \\ 6 & -4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ -3 & 1 & 3 \\ 0 & 2 & y \end{bmatrix}$. If $B = A^{-1}$, then $x + y = ?$

$x(-2) + (-2)(-3) + (-3)(0) = 0 \qquad x = 3$
$6(1) + (-4)(3) + (-7)y = 1 \qquad y = -1$
$x + y = 2$

Answer: C

20. If the line through $(3, 2)$ and $(-4, p)$ is perpendicular to the line $x + 5y = 12$, then $p = ?$

Solve $x + 5y = 12$ for y

$y = -\frac{1}{5}x + \frac{12}{5}$

The slope of the line is $m_1 = -\frac{1}{5}$

The perpendicular slope of m_1 is $m_2 = 5$

Solve $\frac{p-2}{-4-3} = 5$ for p

$p = -33$

Answer: A

21. A merchant sells a radio making a profit equal to 25% of the cost. What is the ratio of the cost to the selling price?

Let p be the price and C the cost

$p = \left(1 + \frac{1}{4}\right)C$

$\frac{C}{p} = \frac{C}{\left(1 + \frac{1}{4}\right)C} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$

Answer: E

22. Find the minimum value of the function $y = \sqrt{2}x^2 + \sqrt{8}x - 2$

$$y = \sqrt{2}(x^2 + 2x) - 2 = \sqrt{2}(x^2 + 2x + 1) - \sqrt{2} - 2 = \sqrt{2}(x+1)^2 - \sqrt{2} - 2$$

When $x = -1$, y has the minimum value $-\sqrt{2} - 2$.

Answer: B

23. (Tie Break No.3) Find the difference between the largest and the smallest values of x that satisfy the equation: $4(8^x) - 21(4^x) + 21(2^x) = 4$

Let $y = 2^x$.

$4y^3 - 21y^2 + 21y - 4 = 0$

$y = 1$ is a solution. Factor out $(y-1)$

$(y-1)(4y^2 - 17y + 4) = 0$

$(y-1)(4y-1)(y-4) = 0$

Then we have $y = 2^x = \frac{1}{4}$, $y = 2^x = 1$, and $y = 2^x = 4$

Corresponding x values are $x = -2$, $x = 0$, and $x = 2$

The difference between the largest and the smallest values of x is 4

Answer: C

24. If $\cos^4 x + \sin^4 x = \frac{5}{9}$ and $p < x < 3p/2$, find the exact value of $\sin 2x$.

$$1 = (\sin^2 x + \cos^2 x)^2 = \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x = \frac{5}{9} + \frac{\sin^2 2x}{2}$$

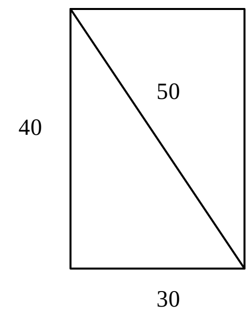
$$\sin^2 2x = \frac{8}{9}$$

Since $2p < 2x < 3p$, we have $\sin 2x = \frac{2\sqrt{2}}{3}$

Answer: A

25. Instead of walking along two adjacent edges of an empty rectangular lot, a boy decides on a shortcut and walks along the diagonal. If the two adjacent edges are 40 and 30 yards respectively, about what percent does he save?

$$\frac{70-50}{70} \approx 0.29 = 29\%$$



Answer: A

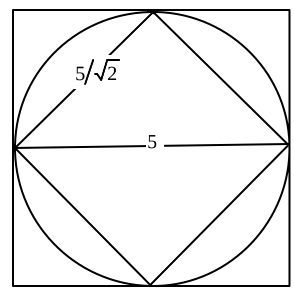
26. Find the area of a square that is inscribed in a circle, which is itself inscribed within a square whose side is 5 inches.

Let x be the length of the side of the small square.

$$x^2 + x^2 = 5^2$$

$$x = \frac{5}{\sqrt{2}}$$

Area of the small square $A = \left(\frac{5}{\sqrt{2}}\right)^2 = 12.5\, in.^2$



Answer: A

27. Solve the equation $\sin 15x + \cos 15x = 0$. What is the sum of the three smallest positive solutions?

Multiply the both sides of the equation by $\sqrt{2}/2$.

$\cos 15x \cdot \left(\frac{\sqrt{2}}{2}\right) + \sin 15x \cdot \left(\frac{\sqrt{2}}{2}\right) = 0$ Rewrite it as $\cos 15x \cos\frac{p}{4} + \sin 15x \sin\frac{p}{4} = 0$

Using the difference formula of cosine $\cos\left(15x - \frac{p}{4}\right) = 0$

$15x - \frac{p}{4} = \frac{p}{2} + 2kp$ or $15x - \frac{p}{4} = \frac{3p}{2} + 2kp$

$x = \frac{3p}{60} + \frac{2kp}{15}\ (k = 0,1,2,\cdots)$ $x = \frac{7p}{60} + \frac{2kp}{15}\ (k = 0,1,2,\cdots)$

$x = \frac{3p}{60}, \frac{11p}{60}, \frac{19p}{60}, \cdots$ $x = \frac{7p}{60}, \frac{15p}{60}, \cdots$

The sum of the three smallest positive solutions is $S = \frac{3p}{60} + \frac{7p}{60} + \frac{11p}{60} = \frac{7p}{20}$

Answer: C

28. Find all values of x in the interval $[0^\circ, 360^\circ]$ that satisfy the equation $\tan 3x + 1 = \sqrt{2}\sec 3x$. The sum of these values is

Square the both sides of the equation $\tan 3x + 1 = \sqrt{2}\sec 3x$

$\tan^2 3x + 2\tan 3x + 1 = 2\sec^2 3x$

$\tan^2 3x + 2\tan 3x + 1 = 2(\tan^2 3x + 1)$

$\tan^2 3x - 2\tan 3x + 1 = 0$

$\tan 3x = 1$

$3x = 45^\circ + k180^\circ,$

$x = 15^\circ + k60^\circ \quad (k = 0,1,2,\cdots,5)$

$x = 15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ,$ or 315°

After checking, $x = 15^\circ, 135^\circ,$ and 255° are the values satisfying the equation.

The sum of these values is $S = 15^\circ + 135^\circ + 255^\circ = 405^\circ$

Answer: D

29. The domain of $f(x) = \log(\sin x)$ contains which of the following intervals?

The domain of $f(x)$ is $\{x \mid \sin x > 0\}$.

The only interval that can be contained in the domain is $0 < x < p$.

Answer: C

30. Point (3, 2) lies on the graph of the inverse of $f(x) = 2x^3 + x + A$. The value of A is

$y = f^{-1}(x) \Leftrightarrow x = f(y)$

$3 = f(2) = 2\cdot(2)^3 + (2) + A$

$3 = 18 + A$

$A = -15$

Answer: B

31. Which of the following is equivalent to $\sin(\tan^{-1} v)$?

Let $q = \tan^{-1} v$, then $\tan q = v$

$$\sin^2 q = \frac{1}{\csc^2 q} = \frac{1}{\cot^2 q + 1} = \frac{\tan^2 q}{1 + \tan^2 q} = \frac{v^2}{1 + v^2}$$

Since the range of $q = \tan^{-1} v$ is $-p/2 < q < p/2$; $\sin q$ and $\tan q$ have the same sign when q is terminated in the quadrant I or IV, we have $\sin q = \dfrac{v}{\sqrt{1 + v^2}}$

Answer: A

32. If $\cos 20^\circ = K$ and $\cos x = 2K^2 - 1$, what are all the possible values of x between 0° and 360° ?

$$\cos x = 2\cos^2 20^\circ - 1 = \cos(2 \cdot 20^\circ) = \cos 40^\circ$$

$x = 40^\circ$ or $x = 360^\circ - 40^\circ = 320^\circ$

Answer: D

33. Find n such that the line $y = x + 8$ is tangent to the graph of the function $f(x) = n\sqrt{x}$.

Let the line be tangent to the curve at (x_1, y_1). $f'(x) = \dfrac{n}{2\sqrt{x}}$ The slope of the line is 1.

$$\frac{n}{2\sqrt{x_1}} = 1 \qquad \textbf{(1)}$$

We also have

$$x_1 + 8 = n\sqrt{x_1} = y_1 \qquad \textbf{(2)}$$

Solve (**1**) for n and substitute the result into (**2**)

$$x_1 + 8 = (2\sqrt{x_1})\sqrt{x_1}$$

$$x_1 = 8$$

$$n = 2\sqrt{8} = 4\sqrt{2}$$

Answer: D

34. (Tie Break No. 4) Solve the equation $\sin^{-1} 2x = p/4 + \sin^{-1} x$ for x.

$\sin(\sin^{-1} 2x) = \sin(p/4 + \sin^{-1} x)$, $\quad 2x = \sin(p/4)\cos(\sin^{-1} x) + \cos(p/4)x$

$2x = \frac{\sqrt{2}}{2}\sqrt{1-x^2} + \frac{\sqrt{2}}{2}x$, $\quad (4-\sqrt{2})x = \sqrt{2}\sqrt{1-x^2}$

Square the both sides of the equation.

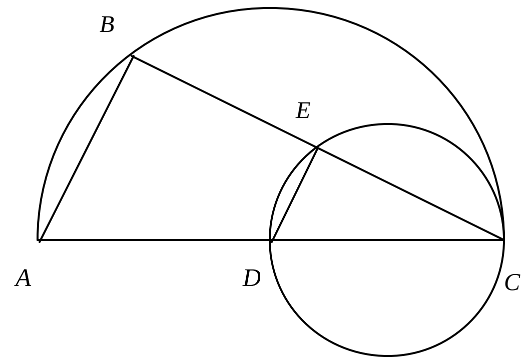
$(16 - 8\sqrt{2} + 2)x^2 = 2 - 2x^2$, then $(20 - 8\sqrt{2})x^2 = 2$

$x^2 = \frac{1}{10-4\sqrt{2}}$, $x = \pm\frac{1}{\sqrt{10-4\sqrt{2}}}$

After checking, $x = \frac{1}{\sqrt{10-4\sqrt{2}}}$ is the solution.

Answer: D

35. Given a circle having center D and a smaller circle having diameter DC, find the ratio of the areas of the triangles ABC and DEC that are inscribed in the large and small semicircle respectively.



ΔABC and ΔDEC are right triangles.

$\Delta ABC \sim \Delta DEC$

$\frac{AC}{DC} = \frac{BC}{EC} = \frac{AB}{DE} = 2$

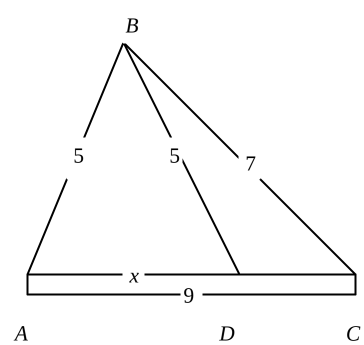
Area of $\Delta ABC = \frac{1}{2}(AB)(BC) =$

$= \frac{1}{2}(2\cdot DE)(2\cdot EC) = 4\left(\frac{1}{2}(DE)(EC)\right) =$

$= 4\,(\text{Area of } \Delta DEC)$

Answer: A

36. In ΔABC, $AB = 5$, $BC = 7$, $AC = 9$, and D is on the side AC such that $BD = 5$. Find the ratio $AD : DC$.



Let $x = AD$. Using the law of cosines, in ΔABD

$\frac{5^2 + x^2 - 5^2}{2\cdot 5\cdot x} = \cos A$ **(1)**

In ΔABC, $\frac{5^2 + 9^2 - 7^2}{2\cdot 5\cdot 9} = \cos A$ **(2)**

Substitute (**2**) into (**1**). We have $\frac{x^2}{10x} = \frac{19}{30}$

$x = \frac{19}{3}$ and $DC = AC - x = 9 - \frac{19}{3} = \frac{8}{3}$

$AC : DC = 19 : 8$

Answer: E

37. A square of perimeter 20 is inscribed in a square of perimeter 28. What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

Using Pythagorean Theorem,

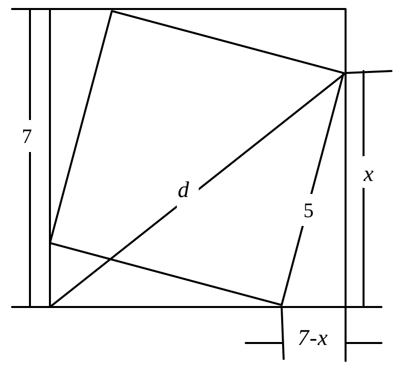
$x^2 + (7-x)^2 = 5^2$

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

Since we need to find greatest distance between a vertex of the inner square and a vertex of the outer square, we use $x = 4$

$d = \sqrt{7^2 + x^2} = \sqrt{49 + 4^2} = \sqrt{65}$



Answer: D

38. In the circle shown, AP is a secant line passing though the center O of the circle; and CP is a tangent line. The length of AP is 8, and the length of CP is 4. Find the length of the diameter *AB*.

Let r be the radius of the circle.
Since ΔOCP is a right triangle, by the Pythagorean Theorem,

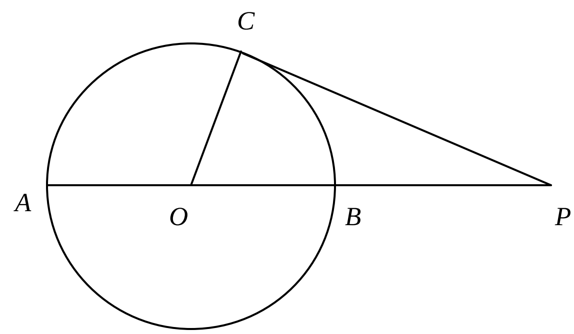
$(OC)^2 + (CP)^2 = (OP)^2$

$r^2 + 4^2 = (8-r)^2$

$r^2 + 16 = 64 - 16r + r^2$

$r = 3$

The diameter $AB = 2r = 6$



Answer: C

39. In the figure, *ABCDEF* is a right prism with triangle base. The altitude of the prism is h. If a plane cuts the figure through points *A*, *C*, and *E*, two solids, *EABC* and *EACFD* are formed. What is the ratio of the volume of *EABC* to the volume of *EACFD*?

Let A be the area of ΔABC

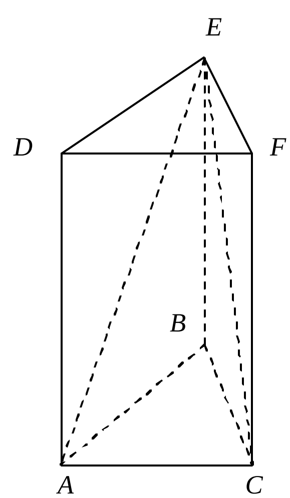
The volume of the solid *EABC* $V_1 = \frac{1}{3}hA$

The volume of the right prism $V = hA$

The volume of the solid *EACFD*

$V_2 = V - V_1 = \frac{2}{3}hA$

The ratio of the volume of *EABC* to the volume of *EACFD* is 1 : 2



Answer: A

40. Judy and Beth planned a 5000 mile trip in a car with 5 tires, 4 on the car and one spare. They plan to rotate the tires so that each of the five tires is on the car for the same number of miles. How many miles will each tire travel?

$$\frac{5000\times 4}{5} = 4000\, miles$$

Answer: B

41. The negation of the statement "For all sets, there is one subset" is

The negation of the statement "For all sets, there is one subset" is "For some sets, there is not one subset."

Answer: C

42. Let x and y be single digit natural numbers. Find the probability that $\frac{x}{y}$ is not equivalent to a natural number.

The cardinality of the sample space is $9^2 = 81$.
The numbers of $\frac{x}{y}$ not equivalent to a natural number are 8, 7, 7, 6, 7, 5, 7, 5, and 6 when x is 1, 2, 3, 4, 5, 6, 7, 8, and 9 respectively.
$8+7+7+6+7+5+7+5+6=58$
Therefore, the probability that $\frac{x}{y}$ is not equivalent to a natural number is $\frac{58}{81}$.

Answer: D

43. The four roots of the equation $(x^2-2x+m)(x^2-2x+n)=0$ form an arithmetic sequence whose first term is $1/4$. Then $|m-n|=?$

Let $\frac{1}{4}, \frac{1}{4}+d, \frac{1}{4}+2d$, and $\frac{1}{4}+3d$ be the four roots of the equation.
Solve $\frac{1}{4}+\left(\frac{1}{4}+3d\right)=\left(\frac{1}{4}+d\right)+\left(\frac{1}{4}+2d\right)=-(-2)$ for d
$d=\frac{1}{2}$

$m=\frac{1}{4}\cdot\left(\frac{1}{4}+3\cdot\frac{1}{2}\right)=\frac{7}{16}$ or $m=\left(\frac{1}{4}+\frac{1}{2}\right)\cdot\left(\frac{1}{4}+2\cdot\frac{1}{2}\right)=\frac{15}{16}$
$n=\frac{15}{16}$ $n=\frac{7}{16}$
$|m-n|=\frac{15}{16}-\frac{7}{16}=\frac{1}{2}$

Answer: B

44. If the function $f(x) = a^x$ is decreasing on $(-\infty,\ \infty)$, and the graph of the function $f(x) = -x^2 + 3ax - 1$ is below the x-axis; we have

The function $f(x) = a^x$ decreasing on $(-\infty,\ \infty)$ implies
$0 < a < 1$ **(1)**
The graph of the function $f(x) = -x^2 + 3ax - 1$ below the x-axis implies the discriminant of the equation $-x^2 + 3ax - 1 = 0$ is less than 0.

$$(3a)^2 - 4 \cdot (-1) \cdot (-1) < 0$$

$$-\frac{2}{3} < a < \frac{2}{3} \qquad \textbf{(2)}$$

The intersection of **(1)** and **(2)** is $0 < a < \frac{2}{3}$

Answer: B

45. (Tie Break No.5) A set of consecutive positive integers beginning with 1 is written on the board. One of the numbers is erased. The average of the remaining numbers is $35\frac{7}{17}$. What number was erased?

The sum of the all numbers in the set is $\frac{n(n+1)}{2}$, where n is the largest number in the set.
We have

$$\frac{n}{2} = \frac{\frac{n(n+1)}{2} - n}{n-1} < 35\frac{7}{17} < \frac{\frac{n(n+1)}{2} - 1}{n-1} = \frac{n+2}{2}$$

This implies $n = 69$, or $n = 70$. Since $n - 1 = 70 - 1 = 69$ is not multiple of 17, it is impossible for $n = 70$. We have $n = 69$
Let x be the number erased. Solve for x

$$\frac{\frac{69(69+1)}{2} - x}{69-1} = 35\frac{7}{17}$$

$$\frac{2415 - x}{68} = \frac{602}{17}$$

$$x = 7$$

Answer: B