Math Bowl 2005
Suggested Solutions to the Written Test

1. Let $f(x) = ax^5 - bx^3 + cx - 10$ If $f(7) = 12$, find $f(-7)$.

$$f(-7) = a(-7)^5 - b(-7)^3 + c(-7) - 10$$
$$= -a \cdot 7^5 + b \cdot 7^3 - c \cdot 7 + 10 - 20$$
$$= -f(7) - 20 = -12 - 20 = -32$$

Answer: (A)

2. If $x^y = 1/2$, then $(x^2)^{3y} = ?$

$$\left(x^2\right)^{3y} = x^{6y} = \left(x^y\right)^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

Answer: (C)

3. If $\log_b 2 = x$, $\log_b 3 = y$, and $\log_b 5 = z$, then $\log_b(0.075) = ?$

$$\log_b 0.075 = \log_b \frac{3}{2^3 \cdot 5} = \log_b 3 - 3\log_b 2 - \log_b 5$$
$$= y - 3x - z$$

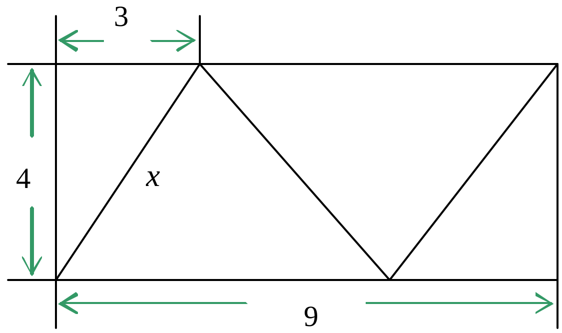
Answer: (B)

4. If the three segments inside the rectangle have the same length, then the sum of their lengths is:

Using the Pythagorean Theorem,

$x = \sqrt{3^2 + 4^2} = 5$

$3x = 15$



Answer: (D)

5. Find $x^2 + y^2$ if x and y are positive integers such that $xy + x + y = 71$ and $x^2y + xy^2 = 880$

We have $\begin{cases} xy + (x+y) = 71 & (1) \\ xy(x+y) = 880 & (2) \end{cases}$.

Substituting $(x+y) = \frac{880}{xy}$ into (1), we have $xy + \frac{880}{xy} = 71$ or $(xy)^2 - 71(xy) + 880 = 0$

$xy = 16$ and $x + y = 55$ or $xy = 55$ and $x + y = 16$. In the first case, x and y are irrational numbers. It contradicts the assumption. In the second case, we have $x = 11$ and $y = 5$, or $x = 5$ and $y = 11$. Then $x^2 + y^2 = 11^2 + 5^2 = 146$

Answer: (A)

6. If $\tan^{-1} 2x + \tan^{-1} 3x = \frac{p}{4}$, what is the value of x ?

$\tan\left(\tan^{-1} 2x + \tan^{-1} 3x\right) = \tan \frac{p}{4}$

$\frac{\tan(\tan^{-1} 2x) + \tan(\tan^{-1} 3x)}{1 - \tan(\tan^{-1} 2x)\tan(\tan^{-1} 3x)} = 1$

$\frac{2x + 3x}{1 - 2x \cdot 3x} = 1$

$6x^2 + 5x - 1 = 0$, $x = \frac{1}{6}$ or $x = -1$

Noticing $\tan^{-1}(-2) + \tan^{-1}(-3) < 0$, $x = -1$ is not the solution. $x = \frac{1}{6}$ is the only solution.

Answer: (E)

7. If $f(2x+1)=4x^2+2x-6$, find the sum of the zeros of $f(x)$.

$f(2x+1)=4x^2+4x+1-2x-1-6=(2x+1)^2-(2x+1)-6$
$f(x)=x^2-x-6$
Since the coefficient of the x^2 term of $f(x)$ is one, the sum of zeros of $f(x)$ is the opposite number of the coefficient of x term. $-(-1)=1$.

Answer: (C)

8. The sum of the first three terms of a geometric sequence of positive integers is equal to seven times the first term. The sum of the first four terms is 45. What is the first term of the sequence?

Let a be the first term of the sequence and r the common ratio.
$a+ar+ar^2=7a$
$a(r^2+r-6)=0$, $r=2$ or $r=-3$. Since the sequence is positive, we only can have $r=2$. $a+ar+ar^2+ar^3=45$, $a(1+2+2^2+2^3)=45$. Hence, $a=3$.

Answer: (C)

9. Three solid balls, each of radius 3.25 *cm*, are stored in a circular cylindrical can with the smallest possible radius and volume. What fraction of the can's volume is air?

Let $r=3.25\ cm$. The radius of the can is r and the height of the can is $6r$.
$V_{can}=\pi r^2\cdot 6r=6\pi r^3$
$V_{3balls}=3\cdot\frac{4}{3}\pi r^3=4\pi r^3$
$\frac{V_{air}}{V_{can}}=\frac{6\pi r^3-4\pi r^3}{6\pi r^3}=\frac{1}{3}$

Answer: (E)

10. Let p be an odd whole number and let n be any whole number. Which of the following statements about the whole number $(p^2 + np)$ is always true?

If n is even, $(p+n)$ is odd since p is odd. Then, the product $p(p+n) = p^2 + np$ is odd. If $(p^2 + np)$ is odd, $(p+n)$ is odd and $n = (p+n) - p$ is even. Therefore, $(p^2 + np)$ is odd if and only if n is even. All the statements are not true except the statement (E).

Answer: (E)

11. If $i^2 = -1$, then $(1-i)^{11} = ?$

$(1-i)^2 = 1 - 2i + i^2 = -2i$

$(1-i)^{11} = \left((1-i)^2\right)^5 (1-i) = (-2i)^5 (1-i) = -32i^5 (1-i) = -32i(1-i)$

$= -32i + 32i^2 = -32 - 32i$

Answer: (B)

12. Given $f(x) = \log \dfrac{1-x}{1+x}$, if $f(a) = b$, then $f(-a) = ?$

$$f(-a) = \log \frac{1+a}{1-a} = \log(1+a) - \log(1-a) = -\left(\log(1-a) - \log(1+a)\right) = -\log \frac{1-a}{1+a} = -b$$

Answer: (B)

13. Which of the following is the solution set of the inequality $1 < |2x+1| < 3$?

The inequality $1 < |2x+1| < 3$ is equivalent to the system of inequalities:

$1 < |2x+1|$ and $|2x+1| < 3$

The solution of $1 < |2x+1|$ is $(-\infty, -1) \cup (0, \infty)$

The solution of $|2x+1| < 3$ is $(-2, 1)$

Their intersection $(-2, -1) \cup (0, 1)$ is the solution of the inequality $1 < |2x+1| < 3$

Answer: (D)

14. If $\sin 2q < 0$ and $\cos q - \sin q < 0$, then q is terminated in which of the following quadrants?

$\sin 2q < 0$ implies $2\sin q \cos q < 0$. $\sin q$ and $\cos q$ have the opposite signs. q is terminated in the quadrant II or IV.

$\cos q - \sin q < 0$ implies $\sin q > \cos q$. We have $\dfrac{p}{4} + 2kp < q < \dfrac{5p}{4} + 2kp$

Therefore, q is terminated in the quadrant II.

Answer: (B)

15. Given $a^2 + b^2 = 1$, $b^2 + c^2 = 2$, and $c^2 + a^2 = 2$, find the minimum value of $ab + bc + ca$

Solve the system $\begin{cases} a^2 + b^2 = 1 \\ b^2 + c^2 = 2 \\ a^2 + c^2 = 2 \end{cases}$

$a^2 = \dfrac{1}{2}$, $b^2 = \dfrac{1}{2}$ and $c^2 = \dfrac{3}{2}$. We have $a = \pm\dfrac{1}{\sqrt{2}}$, $b = \pm\dfrac{1}{\sqrt{2}}$, and $c = \pm\dfrac{\sqrt{3}}{\sqrt{2}}$

After examining all the possibilities, we have when $a = b = \dfrac{1}{\sqrt{2}}$ and $c = -\dfrac{\sqrt{3}}{\sqrt{2}}$ or

$a = b = -\dfrac{1}{\sqrt{2}}$ and $c = \dfrac{\sqrt{3}}{\sqrt{2}}$, $ab + bc + ca$ has the minimum value $\dfrac{1}{2} - \sqrt{3}$

Answer: (E)

16. Find the maximum value of the function $f(x) = \dfrac{1}{1 - x(1-x)}$

$$f(x) = \frac{1}{1-x(1-x)} = \frac{1}{1-x+x^2} = \frac{1}{\frac{3}{4} + \left(\frac{1}{4} - x + x^2\right)} = \frac{1}{\frac{3}{4} + \left(\frac{1}{2} - x\right)^2}$$

When $x = \frac{1}{2}$, the function has the maximum value $f\left(\frac{1}{2}\right) = \frac{4}{3}$

Answer: (A)

17. A sphere is inscribed in a right circular cone. If the height of the cone is 3 times the radius of the sphere, what is the ratio of the volume of the sphere to the volume of the cone?

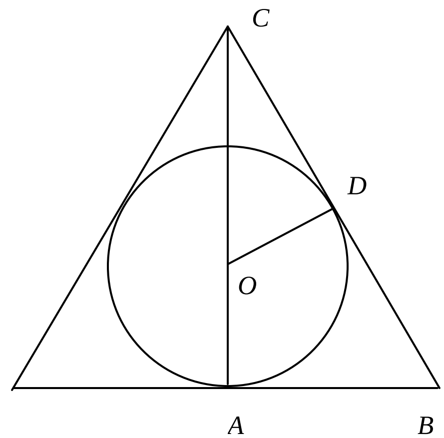
The figure is the intersection of cone and sphere with the plane passing through the axis of the cone.

Let r be the radius of the sphere. $AC = 3r$

$DO = OA = r$, and $OC = 2r$. $OD \perp CD$

$\sin \angle OCD = \frac{OD}{OC} = \frac{1}{2}$ and $\angle OCD = 30^\circ$.

$AB = AC \tan 30^\circ = 3r \cdot \frac{1}{\sqrt{3}} = r\sqrt{3}$



$V_{cone} = \frac{1}{3}p(AB)^2(AC) = \frac{1}{3}p(r\sqrt{3})^2(3r) = 3pr^3$

$V_{sphere} = \frac{4}{3}pr^3$

$V_{sphere} : V_{cone} = 4:9$

Answer: (A)

18. The function $f(x) = x^2 - 2ax - 3$ ($x \in [1, 2]$) has an inverse function, if and only if a satisfies which of the following?

$f(x) = x^2 - 2ax - 3 = (x^2 - 2ax + a^2) - a^2 - 3 = (x-a)^2 - (a^2+3)$

$f(x) = x^2 - 2ax - 3$ ($x \in [1, 2]$) is a one-to-one function if and only if the x- coordinate $x = a$ of the vertex of the parabola $f(x) = x^2 - 2ax - 3$ is not in the interval $(1, 2)$. a has to be in $(-\infty, 1] \cup [2, \infty)$

Answer: (D)

19. If the real numbers a, b, and c satisfy $c < b < a$ and $ac < 0$, Which of following statement is not necessarily true?

$c < b < a$ and $ac < 0$ do not imply $b \neq 0$. If $b = 0$, The statement $cb^2 < ab^2$ is false.

Answer: (C)

20. There are two ways to inscribe a square into an isosceles right triangle. (See the figure (1) and (2).) Find the ratio of the area of the square in the figure (1) to the area of the square in the figure (2) if these two isosceles right triangles are congruent.

Let l be the length of the shorter side of the isosceles right triangle. It is easy to see that the side of the square in the figure (1) is $l/2$. And the area of the square in the figure (1) is $A_1 = l^2/4$.

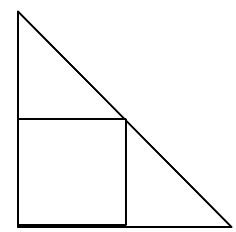


Figure (1)

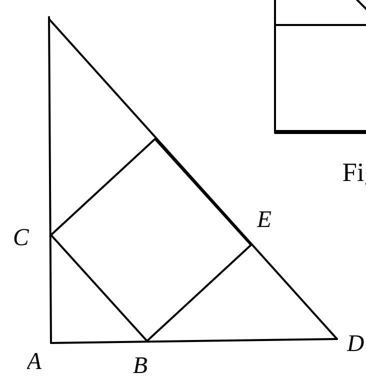


Figure (2)

In the figure (2), let s be the length of the sides of the square. $BC = BE = s$. Noticing the both ΔABC and ΔBDE are $45^\circ - 45^\circ - 90^\circ$ triangles,

We have $AB = BC \cdot \dfrac{\sqrt{2}}{2} = \dfrac{s\sqrt{2}}{2}$ and

$BD = BE \cdot \sqrt{2} = s\sqrt{2}$

$l = AD = AB + BD = \dfrac{3s\sqrt{2}}{2}$ and $s = \dfrac{l\sqrt{2}}{3}$

The area of the square in the figure (2) is $A_2 = \dfrac{2}{9} l^2$

$A_1 : A_2 = \dfrac{1}{4} l^2 : \dfrac{2}{9} l^2 = 9 : 8$

Answer: (B)

21. If a is a real number and in the expansion of $(x + a)^{10}$, the coefficient of x^7 is -15, then $a = ?$

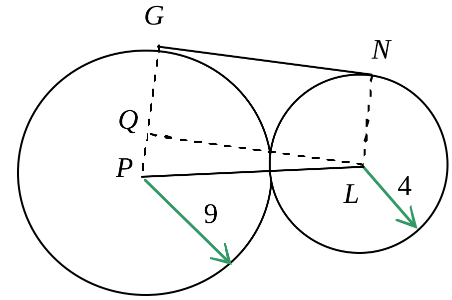
The coefficient of x^7 is $C_{10}^3 a^3 = 120a^3$

$120a^3 = -15$

$a = -\dfrac{1}{2}$

Answer: (D)

22. Circles *P* and *L* are tangent and have radii 9 and 4 respectively. Find the length of the common tangent *GN*.



Connect *PG* and *LN* and draw a line passing through *L* and parallel to *GN*. It intersects *PG* at *Q*. *LQGN* is a rectangle. In the right triangle ΔPQL,

$PL = 9 + 4 = 13$

$PQ = PG - QG = PG - LN = 9 - 4 = 5$

By the Pythagorean Theorem,

$LQ = \sqrt{(PL)^2 - (PQ)^2} = \sqrt{13^2 - 5^2} = 12$

$GN = QL = 12$

Answer: (D)

23. If two marbles are removed at random from a bag containing only black and white marbles, the chance that they are both white is 1/3. If three are removed at random, the chance that all three are white is 1/6. What is the ratio of black balls to white balls?

Let *w* be the number of white balls and *b* the number of black balls in the bag. Then we have $\frac{C_w^2}{C_{w+b}^2} = \frac{1}{3}$ and $\frac{C_w^3}{C_{w+b}^3} = \frac{1}{6}$.

$$\frac{\frac{w(w-1)}{2}}{\frac{(w+b)(w+b-1)}{2}} = \frac{1}{3} \qquad (1)$$

$$\frac{\frac{w(w-1)(w-2)}{6}}{\frac{(w+b)(w+b-1)(w+b-2)}{6}} = \frac{1}{6} \qquad (2)$$

Combining these two expressions, we have $\frac{w-2}{(w-2+b)} = \frac{3}{6}$

Solving for *b*, $b = w - 2$. Substituting it in to (1), we have $\frac{w(w-1)}{(2w-2)(2w-3)} = \frac{1}{3}$

$w = 6$ and $b = 4$. $b : w = 2 : 3$

Answer: (D)

24. Find $i + 2i^2 + 3i^3 + ... + 2004i^{2004} + 2005i^{2005}$

$$i + 2i^2 + 3i^3 + 4i^4 + ... + 2001i^{2001} + 2002i^{2002} + 2003i^{2003} + 2004i^{2004} + 2005i^{2005}$$
$$= ((-2+4) + (-6+8) + \cdots + (-2002+2004)) + ((1-3) + (5-7) + \cdots + (2001-2003) + 2005)i$$
$$= 2\left(\frac{2004}{4}\right) + \left(-2\left(\frac{2003+1}{4}\right) + 2005\right)i = 1002 + 1003i$$

Answer: (B)

25. Express the perimeter of trapezoid $ABCD$ in simplest exact form if $DC = 4$ and $AD = 6$.

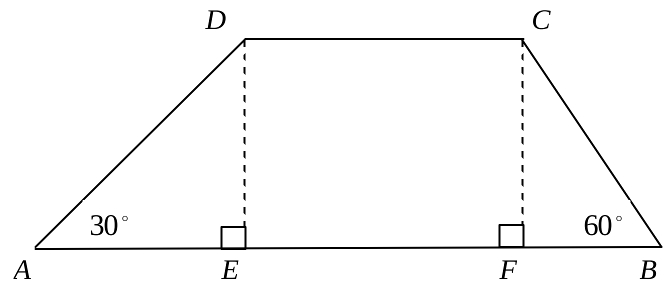
$DE = AD\sin 30^\circ = 6\cdot\left(\frac{1}{2}\right) = 3 = CF$

$AE = AD\cos 30^\circ = 6\cdot\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$

$EF = DC = 4$

$FB = \frac{CF}{\tan 60^\circ} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$CB = \frac{CF}{\sin 60^\circ} = \frac{3}{\frac{\sqrt{3}}{2}} = 2\sqrt{3}$



The perimeter of $ABCD$ is

$AD + AE + EF + FB + BC + DC = 6 + 3\sqrt{3} + 4 + \sqrt{3} + 2\sqrt{3} + 4 = 14 + 6\sqrt{3}$

Answer: (E)

26. What is the solution to the following equation?

$4 = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots + 2^n x^n + \cdots$

$4 = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots + 2^n x^n + \cdots = \frac{1}{1-2x} \quad (|2x| < 1)$

$4(1-2x) = 1$

$x = 3/8$

Answer: (C)

27. A ball is dropped from a height of 24 feet. Each time it drops h feet, it rebounds $\frac{2}{3}h$ feet. Find the total vertical distance traveled by the ball.

The total vertical distance traveled by the ball is the sum

$24 + 24\left(\frac{2}{3}\right) + 24\left(\frac{2}{3}\right) + 24\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right)^2 + \cdots = -24 + 48 + 48\left(\frac{2}{3}\right) + 48\left(\frac{2}{3}\right)^2 + \cdots$

$= -24 + \frac{48}{1-\frac{2}{3}} = 120$

Answer: (C)

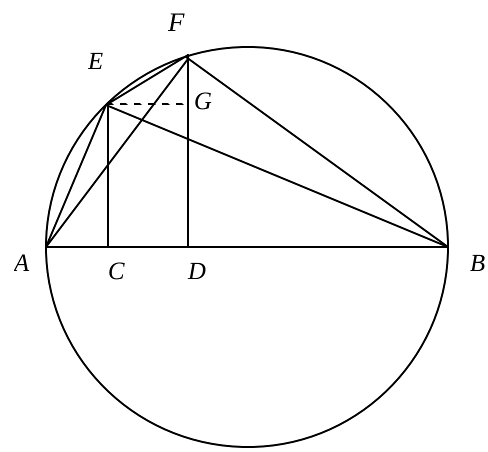
28. If $n \geq 2$ is a natural number, which of the following integers **must** be divisible by 3?

$n(n^2 - 1) = (n-1)n(n+1)$

For any natural number $n \geq 2$, one of $(n-1)$, n, and $(n+1)$ is a multiple of 3.

Answer: (A)

29. Let AB be a diameter of a circle with length 26. Let C and D be located on AB such that $AC = 1$ and $AD = 8$. Let E and F be points on one of the arcs AB for which EC and FD are perpendicular to AB. Find EF.



$DB = AB - AD = 26 - 8 = 18$

Since $\Delta AFD \sim \Delta FBD$, we have $\frac{FD}{AD} = \frac{DB}{FD}$

$(FD)^2 = (AD)(DB) = 8(18) = 144$

$FD = 12$

Similarly, $CB = AB - AC = 26 - 1 = 25$

$\Delta AEC \sim \Delta EBC$, $\frac{EC}{AC} = \frac{CB}{EC}$

$(EC)^2 = (AC)(CB) = 1(25) = 25$

$EC = 5$

Through E, draw a line parallel to AB. It intersects FD at G.

$GF = FD - GD = FD - EC = 12 - 5 = 7$ and $EG = CD = AD - AC = 8 - 1 = 7$

ΔEGF is a right triangle. By Pythagorean Theorem,

$EF = \sqrt{(EG)^2 + (GF)^2} = \sqrt{7^2 + 7^2} = 7\sqrt{2}$

Answer: (D)

30. Find the range of the function $f(x) = \dfrac{6\cos^4 x + 5\sin^2 x - 4}{\cos 2x}$

$$f(x) = \frac{6\cos^4 x + 5(1 - \cos^2 x) - 4}{2\cos^2 x - 1} = \frac{6\cos^4 x - 5\cos^2 x + 1}{2\cos^2 x - 1} = \frac{(3\cos^2 x - 1)(2\cos^2 x - 1)}{2\cos^2 x - 1}$$

Domain of $f(x)$ is $\left\{x \mid x \neq \frac{p}{4} + \frac{kp}{2}\right\}$ (k is any integer)

Since $0 \leq \cos^2 x \leq 1$, $-1 \leq 3\cos^2 x - 1 \leq 2$. We have $3\cos^2\left(\frac{p}{4} + \frac{kp}{2}\right) - 1 = 3\left(\frac{\sqrt{2}}{2}\right)^2 - 1 = \frac{1}{2}$.

Hence, the range of $f(x)$ is $\{y \mid -1 \leq y \leq 2 \, and \, y \neq 1/2\}$ or $\left\{y \mid -1 \leq y < \frac{1}{2} \; or \; \frac{1}{2} < y \leq 2\right\}$

Answer: (C)

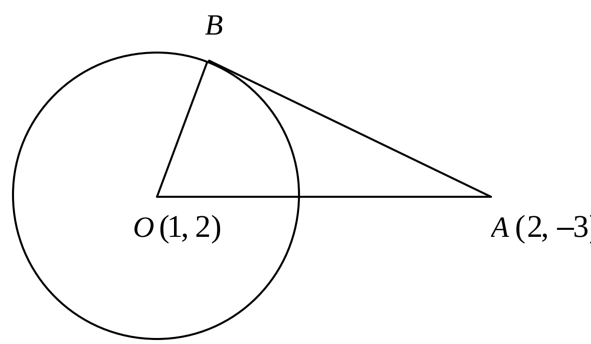
31. Express $\sin 3x$ in terms of $\sin x$.

$$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$$
$$= 2\sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$$
$$= 3\sin x - 4\sin^3 x$$

Answer: (B)

32. If a tangent line is drawn from the point $(2, -3)$ to the circle $x^2 + y^2 - 2x - 4y - 1 = 0$, find the distance from $(2, -3)$ to the point of tangency.

The equation of the circle can be rewritten in
$(x-1)^2 + (y-2)^2 = 6$
It has the center O at $(1, 2)$ and radius $\sqrt{6}$.
The distance from $A\,(2, -3)$ to the center O is
$AO = \sqrt{(2-1)^2 + (-3-2)^2} = \sqrt{26}$
Let B be the point of tangency.
ΔABO is a right triangle.
By the Pythagorean Theorem,
$AB = \sqrt{(AO)^2 - (OB)^2} = \sqrt{26 - 6} = 2\sqrt{5}$



Answer: (C)

33. Find the exact value of $\tan(\cos^{-1}(\sin 30^\circ))$

$$\tan(\cos^{-1}(\sin 30^\circ)) = \tan\left(\cos^{-1}\frac{1}{2}\right) = \tan 60^\circ = \sqrt{3}$$

Answer: (C)

34. Let x be a real number such that $\sec x - \tan x = 2$. Find $\sec x + \tan x$.

$$2 = \frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x} = \frac{\sec^2 x - \tan^2 x}{\sec x + \tan x} = \frac{1}{\sec x + \tan x}$$

$$\sec x + \tan x = \frac{1}{2}$$

Answer: (B)

35. The graph of $f(x) = \cot(4x + f)$ passes through the point $(p/6, 0)$. Which of the following could be a value of f?

$$\cot\left(4\left(\frac{p}{6}\right) + f\right) = 0$$

$\dfrac{2p}{3} + f = \dfrac{p}{2} + kp$ and $f = -\dfrac{p}{6} + kp$, where k is any integer, or $f = \cdots -\dfrac{7p}{6}, -\dfrac{p}{6}, \dfrac{5p}{6}, \cdots$

Answer: (E)

36. Find the minimum value of $y = \sin x + \cos x$.

$$y = \sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = \sqrt{2}\left(\cos\frac{p}{4}\sin x + \sin\frac{p}{4}\cos x\right)$$

$$= \sqrt{2}\sin\left(x + \frac{p}{4}\right)$$

The minimum value of y is $-\sqrt{2}$

Answer: (B)

37. Let $f_n(x) = \sin^n x + \cos^n x$. For how many values of x in $[0, p]$ is it true that $6f_4(x) - 4f_6(x) = 2f_2(x)$?

$$6(\sin^4 x + \cos^4 x) - 4(\sin^6 x + \cos^6 x) = 2(\sin^2 + \cos^2 x)$$
$$6(\sin^4 x + \cos^4 x) - 4(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) = 2$$
$$6(\sin^4 x + \cos^4 x) - 4(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) = 2$$
$$2\sin^4 x + 4\sin^2 x\cos^2 x + 2\cos^4 x = 2$$
$$2(\sin^2 x + \cos^2 x)^2 = 2$$

It is an identity. It is true for any x value.

Answer: (E)

38. Find the following sum: $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+3} - \frac{1}{n+1}\right) = -\frac{1}{2}\left(\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{1}{6} - \frac{1}{4}\right) + \left(\frac{1}{7} - \frac{1}{5}\right) + \cdots\right)$$
$$= -\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{3}\right) = \frac{5}{12}$$

Answer: (D)

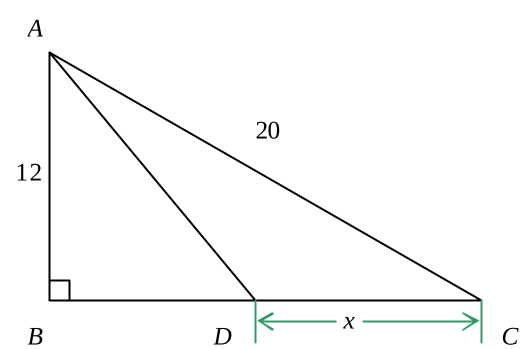
39. In right triangle ABC, if $AD = DB + 8$, what is the value of x?

By the Pythagorean Theorem, in ΔABD

$$12^2 + (DB)^2 = (AD)^2 = (DB + 8)^2$$
$$144 + (DB)^2 = (DB)^2 + 16DB + 64$$
$$DB = 5$$

By the Pythagorean Theorem again, in ΔABC

$$BC = \sqrt{(AC)^2 - (AB)^2} = \sqrt{20^2 - 12^2} = 16$$
$$x = BC - DB = 16 - 5 = 11$$



Answer: (B)

40. Solve the following equation for x: $\sin^{-1} x - \cos^{-1} x = \frac{p}{6}$

$$\sin\left(\sin^{-1} x - \cos^{-1} x\right) = \sin \frac{p}{6}$$

$$\sin(\sin^{-1} x)\cos(\cos^{-1} x) - \cos(\sin^{-1} x)\sin(\cos^{-1} x) = \frac{1}{2}$$

$$x \cdot x - \sqrt{1-x^2} \cdot \sqrt{1-x^2} = \frac{1}{2}$$

$$2x^2 - 1 = \frac{1}{2},\ x^2 = \frac{3}{4},\ x = \pm\frac{\sqrt{3}}{2}$$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{p}{3} - \frac{5p}{6} \neq \frac{p}{6}\ ,\ x = \frac{\sqrt{3}}{2} \text{ is the only solution}$$

Answer: (C)

41. A multiple-choice examination consists of 20 questions. The scoring is $+5$ for each correct answer, -2 for each incorrect answer, and 0 for each unanswered question. John's score is 48. What is the maximum number of questions he could have answered correctly?

Let x be the number of the problems answered correctly and y the number incorrectly.
We have $5x - 2y = 48$. $x = \frac{48+2y}{5}$
Both x and y have to be non-negative integers. $y = 1, 6, 11, \cdots$.
Also, $x + y = 20$. If $y \geq 11$, then $x \geq 14$ and $x + y \geq 20$. We have $y < 11$.
Therefore, when $y = 6$ and x has the maximum value $x = \frac{48 + 2 \cdot 6}{5} = 12$

Answer: (D)

42. Find $\lim_{x\to 0} \frac{\cos 2x - 1}{x^2}$

$$\lim_{x\to 0} \frac{\cos 2x - 1}{x^2} = \lim_{x\to 0} \frac{1 - 2\sin^2 x - 1}{x^2} = \lim_{x\to 0}\left(-2 \cdot \left(\frac{\sin x}{x}\right)^2\right) = -2 \cdot 1^2 = -2$$

Answer: (A)

43. A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk. Ten percent slept through the entire talk. Half of the remainder heard one-third of the talk and the other half heard two-thirds of the talk. What is the average number of minutes of the talk heard by members of the audience?

Let x be the number of the members of the audience.

$$T_{ave} = \frac{60\left(0.2(1)x + 0.1(0)x + \frac{0.7}{2}\left(\frac{1}{3}\right)x + \frac{0.7}{2}\left(\frac{2}{3}\right)x\right)}{x} = 60(0.55) = 33$$

Answer: (D)

44. An international mathematics conference was held at a neutral site. A total of 15 delegates were from Japan, England, United States, and Russia. Each country sent a different number of delegates, and each was represented by at least one delegate. The United States and England sent a total of 6 delegates. One country sent exactly four delegates. England and Russia sent a total of 7 delegates. Which country sent the most number of delegates?

Since one country sent exactly four delegates, we can list the all possibilities as the following:

	England	Japan	Russia	United States
No.1	4	6	3	2
No.2	2	4	5	4
No.3	3	5	4	3
No.4	2	4	5	4

Since each country sent a different number of delegates. The only possibility is the case No. 1.

Answer: (B)

45. The areas of two similar triangles are 45 cm^2 and 80 cm^2. The sum of their perimeters is 35 cm. Find the perimeters of the triangles.

Let A_i and P_i $(i = 1, 2)$ be the areas and perimeters of the similar triangles.

We have $\frac{45}{80} = \frac{A_1}{A_2} = \frac{P_1^{\ 2}}{P_2^{\ 2}}$, $16P_1^{\ 2} = 9P_2^{\ 2}$,

Solve the system $\begin{cases} 4P_1 = 3P_2 \\ P_1 + P_2 = 35 \end{cases}$. $P_1 = 15$ and $P_2 = 20$

Answer: (A)