Math Bowl 2006
Suggested Solutions to the Written Test

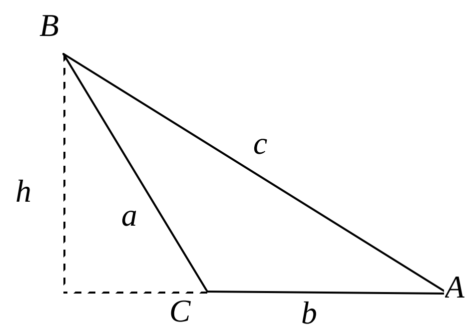
1. If $xy = -3$ and $x^2 + y^2 = 12$, then $(x-y)^2 = ?$

$$(x-y)^2 = x^2 + y^2 - 2xy = 12 - 2(-3) = 18$$

Answer: B

2. In ΔABC, $m\angle C = 120°$, $a = 3$, and $b = 5$. Find $m\angle A$

$$c = \sqrt{a^2 + b^2 - 2ab\cos C}$$
$$= \sqrt{3^2 + 5^2 - 2\cdot 3\cdot 5\cos 120^\circ} = 7$$
$$h = a\sin(180^\circ - C) = \frac{3\sqrt{3}}{2}$$
$$\sin A = \frac{h}{c} = \frac{\frac{3\sqrt{3}}{2}}{7} = \frac{3\sqrt{3}}{14}$$
$$A = \sin^{-1}\left(\frac{3\sqrt{3}}{14}\right)$$



Answer: C

3. A regular hexagon and an equilateral triangle have the same perimeter. What is the ratio of the area of the hexagon to the area of the triangle?

Let a be the length of the side of the equilateral triangle.

The area of the triangle is $A_T = \frac{1}{2}a^2 \sin 60^\circ = \frac{a^2\sqrt{3}}{4}$

The length of the side of the regular hexagon is $\frac{a}{2}$

The area of the hexagon is $A_H = 6\cdot\left(\frac{1}{2}\left(\frac{a}{2}\right)^2 \sin 60^\circ\right) = \frac{3a^2\sqrt{3}}{8}$

$A_H : A_T = 3:2$

Answer: C

4. The period of a simple pendulum is directly proportional to the square root of its length. If a pendulum has a length of 6 feet and a period of 2 seconds, to what length should it be shortened to achieve a 1 second period?

Let p be the period of the pendulum and l the length of the pendulum.

$p = k\sqrt{l}$ and $2 = k\sqrt{6}$

$k = \frac{2}{\sqrt{6}}$

When $p = 1$, $\sqrt{l} = \frac{1}{\frac{2}{\sqrt{6}}} = \frac{\sqrt{6}}{2}$ and $l = \frac{6}{4} = 1.5$

Answer: B

5. If $x^2 - x - 1$ divides $ax^6 + bx^5 + 1$ evenly, find the sum of a and b.

The remainder of $(ax^6 + bx^5 + 1)$ divided by $(x^2 - x - 1)$ is $(8a + 5b)x + (5a + 3b + 1)$

We have to have

$$\begin{cases} 8a + 5b = 0 \\ 5a + 3b = -1 \end{cases}$$

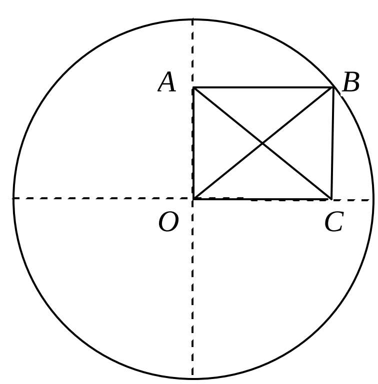
Solve the system

$a = -5$, $b = 8$, and $a + b = 3$

Answer: A

6. Given a circle of radius r with a rectangle $ABCO$ inscribed in one quadrant, what is the length of diagonal AC?

$AC = BO = r$



Answer: E

7. The number of horses and sheep in a certain town is such that the ratio of the difference to the sum is 1:7 and the ratio of the sum to the product is 7:24. What is the sum of the number of horses and sheep?

Let x be the number of the horses and y the number of the sheep.

$$\begin{cases} \dfrac{x-y}{x+y} = \dfrac{1}{7} \\ \dfrac{x+y}{xy} = \dfrac{7}{24} \end{cases} \text{ or } \begin{cases} 6x - 8y = 0 \\ 24x + 24y = 7xy \end{cases}$$

Solving the system, we have $x = 8$, $y = 6$, and $x + y = 14$

Answer: B

8. Tom can plow a field in 12 hours, but with Pat helping him they can plow the field together in 8 hours. If Pat works alone plowing for 12 hours, how long will it take Tom working alone to plow the remainder of the field?

Let x be the hours Pat needs to work alone.

$$\frac{1}{12} + \frac{1}{x} = \frac{1}{8}$$

$$x = 24$$

The hours to takes Tom working alone to finish the remainder is

$$\frac{1 - 12 \cdot \frac{1}{24}}{\frac{1}{12}} = 6$$

Answer: D

9. The will of an eccentric millionaire reads as follows: "I leave $4/17$ of my estate to my son, $7/13$ of the remainder to my wife, $2/3$ of this remainder to my daughter, and the remaining \$2,000,000 to my dog." What was the total amount of the estate?

Let x be the total amount of the estate. $\left(x - \frac{4}{17}x\right) \cdot \frac{7}{13} = \frac{7}{17}x$ will be to his wife. $\left(x - \frac{4}{17}x - \frac{7}{17}x\right) \cdot \frac{2}{3} = \frac{4}{17}x$ will be to his daughter. $x - \frac{4}{17}x - \frac{7}{17}x - \frac{4}{17}x = 2{,}000{,}000$ will be to his dog.. Solving the equation, we have $x = 17{,}000{,}000$

Answer: E

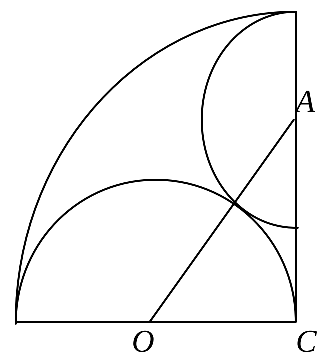
10. A survey of 40 shoppers found that 12 liked Brand X, 23 liked Brand Y, and twice as many disliked both brands as liked both brands. How many liked both brands?

Let x be the number who liked both brands. Then $2x$ is the number who disliked both.
We have $12 + 23 - x + 2x = 40$.
Solving the equation, $x = 5$

Answer: D

11.**(Tie Break No.1)** The radius of the quarter-circle is 2. The two semi-circles are tangent to each other. Find the radius of the smaller semicircle.

Let x be the radius of the small semicircle, O the center of the large semicircle, and A the center of the small semicircle.
$OC = 1$, $AC = 2 - x$, and $AO = 1 + x$
$1^2 + (2 - x)^2 = (1 + x)^2$
Solving the equation, we have
$x = \frac{2}{3}$



Answer: B

12. In a right triangle ΔABC, $C = 90^\circ$. Which of the following statements is NOT true?

$\sin A + \sin B = \sin A + \sin(90^\circ - A) = \sin A + \cos A = \sqrt{2}(\sin A \cos 45^\circ + \cos A \sin 45^\circ)$
$= \sqrt{2}\sin(A + 45^\circ)$
$0 < \sin(A + 45^\circ) \le 1$ and $0 < \sin A + \sin B \le \sqrt{2}$
The statement A is true.
$\sin^2 A + \sin^2 B = \sin^2 A + \cos^2 A = 1 = \sin^2 90^\circ = \sin^2 C$
The statement B is true.
$\cos^2 A + \cos^2 B = \cos^2 A + \sin^2 A = 1$
The statement C is true.
$\tan\frac{A + B}{2} = \tan 45^\circ = 1 = \sin 90^\circ = \sin C$
The statement D is true.

Answer: E

13.**(Tie Break No.2)** What is the value of the expression

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+...+\frac{1}{\log_{100} 100!}?$$

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$$
$$=\frac{\log_2 2}{\log_2 100!}+\frac{\log_3 3}{\log_3 100!}+\frac{\log_4 4}{\log_4 100!}+\cdots+\frac{\log_{100} 100}{\log_{100} 100!}$$
$$=\log_{100!} 2+\log_{100!} 3+\log_{100!} 4+\cdots+\log_{100!} 100$$
$$=\log_{100!} 100!=1$$

Answer: A

14. If $x^2=x+5$, then $x^3=?$

$$x^3=x(x+5)=x^2+5x=(x+5)+5x=6x+5$$

Answer: A

15. If the square of the geometric mean of x and y is equal to the sum of x and y, express y in terms of x.

$$\left(\sqrt{xy}\right)^2=x+y$$

$$xy-y=x$$
$$y=\frac{x}{x-1}$$

Answer: B

16.**(Tie Break No.3)** The library in Destin has between 1000 and 2000 books. Of these, exactly 25% are fiction, $\frac{1}{13}$ are biographies, and $\frac{1}{17}$ are atlases. How many books are either biographies or atlases?

The total number of the books should be a multiple of $4 \cdot 13 \cdot 17 = 884$. Only $2 \cdot 884 = 1768$ is between 1000 and 2000. The number of either biographies or atlases is $\frac{1768}{13} + \frac{1768}{17} = 240$

Answer: A

17. If n is an integer, which of the following must be true?

$3n+1$ is not odd if n is odd. The statement A is not always true
$n(n+2)$ is not even if n is odd. The statement B is not always true
$n(n+1)$ is not divisible by 3 if $n=4$. The statement D is not always true
$n(n+3)$ is not divisible by 4 if $n=6$. The statement E is not always true
$n(3n+3) = 3n(n+1)$ and one of n and $(n+1)$ must be even. $n(3n+3)$ is divisible by 6.

Answer: C

18. $x(x(x(x+2)+2)+2)+2 = ?$

$$x(x(x(x+2)+2)+2)+2 = x(x(x^2+2x+2)+2)+2 = x(x^3+2x^2+2x+2)+2$$
$$= x^4+2x^3+2x^2+2x+2$$

Answer: A

19. Let $ABCD$ be a square piece of paper. A is folded onto C then B is folded onto D. The area of the resulting figure is 9 square inches. Find the perimeter of the square $ABCD$.

Let d be the length of the diagonal of the square. After folding twice, the figure is an isosceles right triangle. Its area is $\frac{1}{2}\left(\frac{d}{2}\right)^2 = 9$ and $d = 6\sqrt{2}$

The length of the side of the square is 6 and the perimeter of the square is 24.

Answer: D

20. If $f(x) = \sqrt[3]{x}$ and $g(x) = x - 5$, find $\left(f^{-1} \circ g^{-1}\right)(-3)$

$f^{-1}(x) = x^3$ and $g^{-1}(x) = x + 5$

$\left(f^{-1} \circ g^{-1}\right)(x) = (x+5)^3$

$\left(f^{-1} \circ g^{-1}\right)(-3) = (-3+5)^3 = 8$

Answer: E

21. If a 1-inch thick slab is sliced off one end of a cube, 294 cubic inches remain. What is the length of an edge of this cube?

Let x be the length of the edge of the cube.

$x^2(x-1) = 294$

$x^3 - x^2 - 294 = 0$

Noticing that 7 is a factor of 294, we can use grouping method.

$(x^3 - 7x^2) + (6x^2 - 42x) + (42x - 294) = 0$

$(x-7)(x^2 + 6x + 42) = 0$

$x = 7$

Answer: C

22. Let $A=\begin{bmatrix}1 & 1 & z\\ 0 & 1 & 0\\ x & -1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}3 & -4 & -1\\ 0 & y & 0\\ -2 & 3 & 1\end{bmatrix}$, find the values of x, y, and z so that $B=A^{-1}$. Then $x+y+z=$?

$$AB=\begin{bmatrix}3-2z & -4+y+3z & -1+z\\ 0 & y & 0\\ 3x-6 & -4x-y+9 & -x+3\end{bmatrix}=I$$

$3-2z=1$, $y=1$, and $-x+3=1$

$x=2$, $y=1$, and $z=1$

$x+y+z=4$

Answer: D

23. In the figure, it is given that angle C = 90°, $\overline{AD}=\overline{DB}$, DE perpendicular to AB, $\overline{AB}=20$, and $\overline{AC}=12$. Find the area of the quadrilateral ADEC.

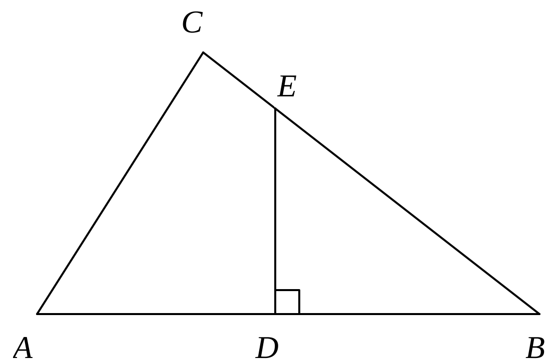
$\overline{AD}=\overline{DB}=10$, $\overline{BC}=\sqrt{(\overline{AB})^2-(\overline{AC})^2}=16$

$\Delta ABC \sim \Delta EBD$

$\overline{ED}=\frac{15}{2}$, $\overline{EB}=\frac{25}{2}$, and $\overline{CE}=16-\frac{25}{2}=\frac{7}{2}$

Area of the quadrilateral is

$\frac{1}{2}\overline{AC}\cdot\overline{CE}+\frac{1}{2}\overline{AD}\cdot\overline{ED}=21+37.5=58.5$



Answer: B

24. Find the exact value of $\tan\left(\cos^{-1}\frac{1}{3}-\sin^{-1}\frac{1}{3}\right)$

Let $A=\cos^{-1}\frac{1}{3}$, then $\tan A=2\sqrt{2}$. Let $B=\sin^{-1}\frac{1}{3}$, then $\tan B=\frac{1}{2\sqrt{2}}$

$$\tan\left(\cos^{-1}\frac{1}{3}-\sin^{-1}\frac{1}{3}\right)=\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$=\frac{2\sqrt{2}-\frac{1}{2\sqrt{2}}}{1+1}=\frac{7}{4\sqrt{2}}$$

Answer: B

25. The sum of the solutions of $(\log_4 x)^2 = \log_4(x^2)$ is equal to?

$(\log_4 x)^2 - 2\log_4 x = 0$
$(\log_4 x)(\log_4 x - 2) = 0$
$\log_4 x = 0$, $x = 1$ and $\log_4 x = 2$, $x = 4^2 = 16$
The sum of the solutions is 17.

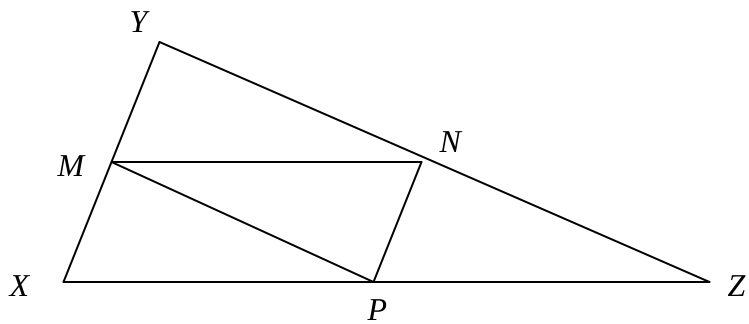
Answer: E

26. In ΔXYZ, points *M*, *N*, and *P* are midpoints. If $\overline{XY} = 10$, $\overline{YZ} = 15$, and $\overline{XZ} = 17$, what is the perimeter of ΔMNP?

$\overline{NP} = \frac{1}{2}\overline{XY} = 5$

$\overline{MP} = \frac{1}{2}\overline{YZ} = \frac{15}{2}$

$\overline{MN} = \frac{1}{2}\overline{XZ} = \frac{17}{2}$



The perimeter of ΔMNP is 21

Answer: D

27. If $f(x^5) = \log_a x$, then $f(2) = ?$

$$f(2) = f\left(\left(2^{\frac{1}{5}}\right)^5\right) = \log_a 2^{\frac{1}{5}} = \frac{\log_a 2}{5}$$

Answer: D

28. Find $3a+2b+c$ if the parabola $y=ax^2+bx+c$ passes through the points $(1,-2)$, $(-2,19)$, and $(3,4)$.

Solve the system of linear equations

$$\begin{cases} a+b+c=-2 \\ 4a-2b+c=19 \\ 9a+3b+c=4 \end{cases}$$

$a=2$, $b=-5$, $c=1$, and $3a+2b+c=-3$

Answer: E

29. Simplify $\left(\frac{(x+1)^2(x^2-x+1)^2}{x^3+1}\right)^2\cdot\left(\frac{(x-1)^2(x^2+x+1)^2}{x^3-1}\right)^2$

$$\left(\frac{(x+1)^2(x^2-x+1)^2}{x^3+1}\right)^2\cdot\left(\frac{(x-1)^2(x^2+x+1)^2}{x^3-1}\right)^2=\left(\frac{(x^3+1)^2}{x^3+1}\right)^2\cdot\left(\frac{(x^3-1)^2}{x^3-1}\right)^2$$
$$=(x^3+1)^2(x^3-1)^2=(x^6-1)^2$$

Answer: D

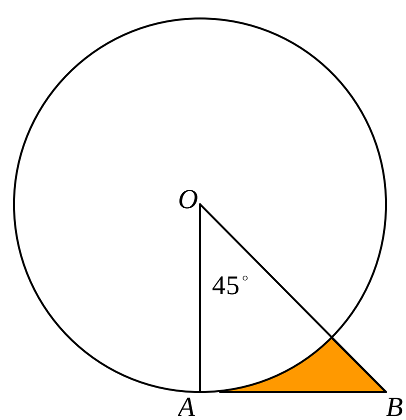
30. In the figure, if the radius OA of the circle is 6 and AB is tangent to the circle what is the area of the shaded part?

The area of the triangle: $A_1=\frac{1}{2}(6^2)=18$

The area of the sector: $A_2=\frac{45^\circ}{360^\circ}(6^2)p=\frac{9}{2}p$

The area of the shaded part

$A=A_1-A_2=18-\frac{9}{2}p$



Answer: B

31. It is given that $\left(r+\frac{1}{r}\right)^2=3$. Find the value of $r^3+\frac{1}{r^3}$

Expanding $\left(r+\frac{1}{r}\right)^2=3$, we have $r^2+2+\frac{1}{r^2}=3$ that implies $r^2-1+\frac{1}{r^2}=0$

$$r^3+\frac{1}{r^3}=\left(r+\frac{1}{r}\right)\left(r^2-1+\frac{1}{r^2}\right)=\left(r+\frac{1}{r}\right)\cdot 0=0$$

Answer: C

32. A rectangular box with a square base has the property that the ratio of its volume to its surface area is 1. If one side of the square base has length a units, express the height b in terms of a.

Let b be the height of the box. The volume of the box is $V=a^2b$
The surface area of the box is $S=2a^2+4ab$

$$\frac{V}{S}=\frac{a^2b}{2a^2+4ab}=1$$

$$b=\frac{2a^2}{a^2-4a}=\frac{2a}{a-4}$$

Answer: A

33. Let $F(x)=\sqrt{\sin^4 x+4\cos^2 x}-\sqrt{\cos^4 x+4\sin^2 x}$. Which of the following expressions is equivalent to $F(x)$?

$$F(x)=\sqrt{\sin x^4+4(1-\sin^2 x)}-\sqrt{\cos^4 x+4(1-\cos^2 x)}$$
$$=\sqrt{(2-\sin^2 x)^2}-\sqrt{(2-\cos^2 x)^2}$$
$$=2-\sin^2 x-2+\cos^2 x=\cos 2x$$

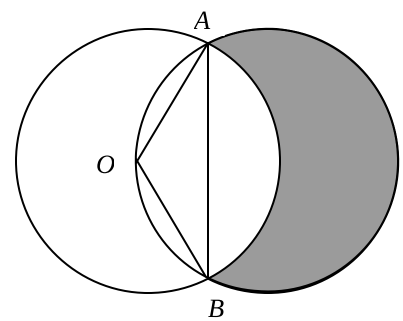
Answer: D

34. Find the exact value of $\tan\left(\frac{5p}{12}\right)$.

$$\tan\left(\frac{5p}{12}\right)=\tan\left(\frac{1}{2}\left(\frac{5p}{6}\right)\right)=\frac{1-\cos\frac{5p}{6}}{\sin\frac{5p}{6}}=\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}=2+\sqrt{3}$$

Answer: E

35.**(Tie Break No.4)** In the figure, two circles have the same radius 6 and the center of one circle is on the other one. Find the area of the shaded region.



$m\angle AOB=120^\circ$

The area of the sector AOB: $A_1=\frac{120^\circ}{360^\circ}(6^2)p=12p$

The area of the triangle ΔAOB :

$A_2=\frac{1}{2}(6^2)\sin 120^\circ=9\sqrt{3}$

The area of the overlapped region of the two circles:

$A_3=2(A_1-A_2)=24p-18\sqrt{3}$

The area of the shaded region: $A=p(6^2)-A_3=36p-(24p-18\sqrt{3})$

$A=12p+18\sqrt{3}$

Answer: C

36. How many ordered triples of positive integers (x,y,z) satisfy $\left(x^y\right)^z=64$

$$\left(2^1\right)^6=\left(2^6\right)^1=\left(2^2\right)^3=\left(2^3\right)^2=\left(4^1\right)^3=\left(4^3\right)^1=\left(8^1\right)^2=\left(8^2\right)^1=\left(64^1\right)^1=64$$

Answer: D

37. Given $r = \dfrac{3}{4\cos q - \sin q}$, find an equivalent equation in rectangular coordinates.

$$r(4\cos q - \sin q) = 3$$
$$4r\cos q - r\sin q = 3$$
$$4x - y = 3$$

Answer: B

38.**(Tie Break No.5)** Find the minimum value of the function $f(x) = \dfrac{1 - \cos 2x + 4\cos^2 x}{\sin 2x}$ on the open interval $(0, p/2)$

Use the fact that $a + b \ge 2\sqrt{ab}$ if $a \ge 0$ and $b \ge 0$ and when $a = b$, the equality holds.

$$f(x) = \frac{1 - \cos 2x + 4\cos^2 x}{\sin 2x} = \frac{1 - (1 - 2\sin^2 x)}{2\sin x\cos x} + \frac{4\cos^2 x}{2\sin x\cos x}$$
$$= \tan x + 2\cot x \ge 2\sqrt{(\tan x)\cdot(2\cot x)} = 2\sqrt{2}$$

The function has the minimum value $2\sqrt{2}$ when $\tan x = \sqrt{2}$

Answer: B

39. $\sin^{-1}(\sin(p + 1)) =$

Noticing that $\sin(p + x) = -\sin x$, $y = \sin^{-1} x$ being an odd function, and $0 < 1 < \dfrac{p}{2}$, we have

$$\sin^{-1}(\sin(p + 1)) = \sin^{-1}(-\sin 1) = -\sin^{-1}(\sin 1) = -1$$

Answer: B

40. If $x = 2\sin\theta$ and $0 \le \theta < \dfrac{\pi}{2}$, then $\ln|\sec\theta + \tan\theta| - \ln|\cos\theta| = ?$

$$\ln|\sec\theta + \tan\theta| - \ln|\cos\theta| = \ln\left|\frac{\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\cos\theta}\right| = \ln\left|\frac{1+\sin\theta}{\cos^2\theta}\right|$$

$$= \ln\left|\frac{1+\sin\theta}{1-\sin^2\theta}\right| = \ln\left|\frac{1}{1-\sin\theta}\right| = \ln\left|\frac{2}{2-x}\right|$$

Answer: D

41. Let $F(x) = g(3x^2)$ and $g(x)$ be a differentiable function. If $g'(12) = 5$ and $g(12) = -3$, find $F'(2)$.

$$F'(x) = g'(3x^2)\cdot 6x$$
$$F'(2) = g'(12)\cdot 12 = 5\cdot 12 = 60$$

Answer: C

42. If $\cos x = \dfrac{3}{5}$ and $\cot x$ is negative, find the value of $\dfrac{\sin x - \tan x}{1+\sec x}$.

$\sin x = -\dfrac{4}{5}$, $\tan x = -\dfrac{4}{3}$, and $\sec x = \dfrac{5}{3}$

$$\frac{\sin x - \tan x}{1+\sec x} = \frac{-\dfrac{4}{5}+\dfrac{4}{3}}{1+\dfrac{5}{3}} = \frac{1}{5}$$

Answer: B

43. The oracle of Niceville tells the truth whenever she chooses to answer a question, and the probability that she will choose to answer a question is 1/3. A student plans to ask the oracle 3 times for his grade on the semester exam. What is the probability that he will get the answer?

The probability that he will get the answer is

$$1 - P\text{ (no answer)} = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$$

Answer: B

44. The perimeter of a rectangle is 100 and its diagonal has length x. What is the area of the rectangle?

Let a be the length of one side of the rectangle. The length of the other side is $50 - a$

$$a^2 + (50 - a)^2 = x^2$$

$$2a^2 - 100a + 2500 = x^2$$

$$2500 - x^2 = 100a - 2a^2 = 2a(50 - a)$$

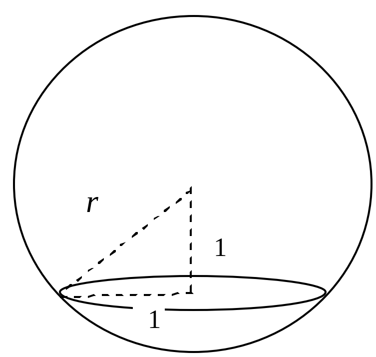
The area of the rectangle is $a(50 - a) = 1250 - \dfrac{x^2}{2}$

Answer: D

45. The area of the circle formed by the intersection of a sphere and a plane is p . If the distance from the center of the sphere to the plane is 1, find the surface area of the sphere.

The radius of the circle is 1 and the radius of the sphere is $r = \sqrt{1^2 + 1^2} = \sqrt{2}$
The surface area of the sphere is

$$S = 4p\, r^2 = 4p\left(\sqrt{2}\right)^2 = 8p$$



Answer: C