Suggested Solutions of Written Test

1. If the average of two numbers is $2x+1$ and one of the numbers is x, what is the other number?
Let y be the other number.

$$\frac{x+y}{2}=2x+1$$
$$x+y=4x+2$$
$$y=3x+2$$

Answer: D

2. Find the perimeter of a regular hexagon inscribed in a circle of radius 5 meters.

The side of the hexagon has length 5 meters. The perimeter of the regular hexagon is $5\times 6=30$ meters.

Answer: A

3. The two shortest sides of a right triangle have lengths 2 and $\sqrt{5}$ respectively. Let x be the smallest angle of the triangle. What is $\cos x$?

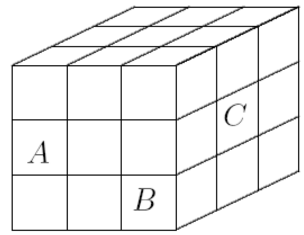
The hypotenuse of the right triangle has length $\sqrt{2^2+\left(\sqrt{5}\right)^2}=3$. The adjacent side of the smallest angle x has length $\sqrt{5}$.

$$\cos x=\frac{\sqrt{5}}{3}$$

Answer: D

4. The figure below shows a solid cube with edge length 3. What is the surface area of the figure obtained by removing the three labeled unit cubes from the large cube shown?

The surface area of the large cube is 54. The surface area is increased by 2 after the unit cube *A* is removed. The surface area is unchanged after the unit cube *B* is removed. And the surface area is increased by 4 after the unit cube *C* is removed. Therefore, the surface area of the figure is $54+2+4=60$ after the three labeled unit cubes are removed



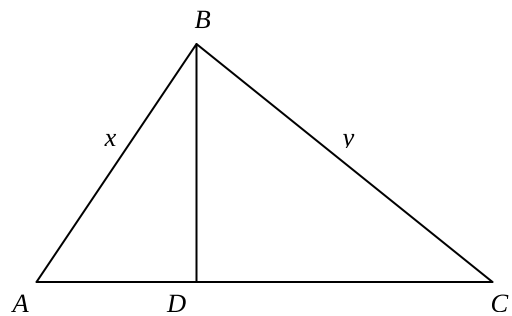
Answer: C

5. In the figure, triangle *ABC* is a right triangle with right angle *B*. *BD* is an altitude of the triangle. If $AB = x$ and $BC = y$, find the ratio of the area of triangle *ABD* to the area of triangle *BCD*.

ΔABD is similar to ΔBCD

$$\frac{AD}{BD}=\frac{AB}{BC}=\frac{x}{y} \text{ and } \frac{BD}{DC}=\frac{AB}{BC}=\frac{x}{y}$$

$$\frac{area\,of\ \Delta ABD}{area\,of\ \Delta BCD}=\frac{\frac{1}{2}(AD)(BD)}{\frac{1}{2}(BD)(DC)}=\frac{x^2}{y^2}$$



Answer: B

6. Hurricanes are very low pressure areas with diameters of over 500 miles. The barometric air pressure "*P*" in inches of mercury at a distance of "*x*" miles from the eye of a hurricane is modeled by the function $P(x) = 0.5\log_9(x+1)+27$. At what distance from the eye of the hurricane is the air pressure 28 inches of mercury?

$$28 = 0.5\log_9(x+1)+27$$
$$1 = 0.5\log_9(x+1)$$
$$2 = \log_9(x+1)$$
$$9^2 = x+1$$
$$x = 80$$

Answer: C

7. Let P be the point $(3, 2)$. Let Q be the reflection of P about the x-axis. Let R be the reflection of Q about the line $y = -x$, and let S be the reflection of R about the origin. What is the distance between points P and S?

The point Q is $(3, -2)$, the point R is $(2, -3)$, and the point S is $(-2, 3)$
The distance between P and S is

$$\sqrt{(3-(-2))^2 + (2-3)^2} = \sqrt{26}$$

Answer: B

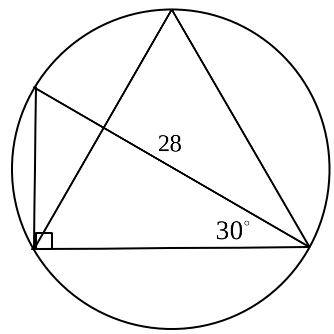
8. The diameter of a circle is 28. An equilateral triangle is inscribed in the circle. What is the perimeter of the triangle?

Let the diameter bisect one of the angles as shown in the figure.
The side of the triangle has length

$$28\cos 30^\circ = 14\sqrt{3}$$

The perimeter of the triangle is

$$3 \times 14\sqrt{3} = 42\sqrt{3}$$



Answer: A

9. A cup of coffee and a cup of cream are sitting side by side on a counter, and they contain equal volumes of liquid. Bob removes $1/10$ of the cream from its cup and adds it to the cup of coffee. Then he removes $1/10$ of this mixture and adds it to the cup of cream. What portion of the mixture in the cream cup is coffee?

Let V represent the initial volume of liquid in each cup. After $\frac{1}{10}$ of the cream is removed and added to the coffee, the volume of $\frac{1}{10}$ of the mixture in the coffee cup is $\frac{11}{100}V$ and contains $\frac{1}{10}V$ coffee. After the mixture is added into the cream cup, the volume of the liquid in the cream cup is $\frac{11}{100}V + \frac{9}{10}V$. The coffee portion in the cream cup is

$$\frac{\frac{1}{10}V}{\frac{11}{100}V + \frac{9}{10}V} = \frac{10}{101}$$

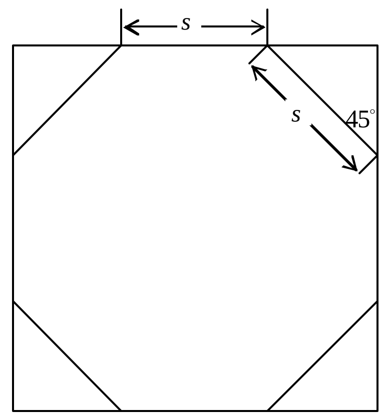
Answer: D

10. A regular octagon stop sign is to be made from a square piece of sheet metal. The largest stop sign that can be cut from the square has sides of length s. What is the area of the square?

The side of the square is $s + 2s\sin 45^\circ = \left(1+\sqrt{2}\right)s$

The area of the square is

$$\left(1+\sqrt{2}\right)^2 s^2 = \left(3+2\sqrt{2}\right)s^2$$



Answer: D

11. The volume "V" of a sphere with radius "r" is given by $V(r) = \frac{4}{3}pr^3$, and the surface area "S" is given by $S(r) = 4pr^2$. Find $V(S)$.

$$r = \left(\frac{S}{4p}\right)^{\frac{1}{2}}$$

$$V(S) = \frac{4}{3}p\left(\frac{S}{4p}\right)^{\frac{3}{2}} = \frac{p}{6}\left(\frac{S}{p}\right)^{\frac{3}{2}}$$

Answer: D

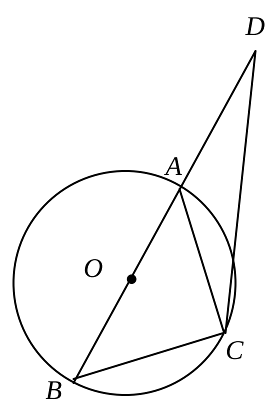
12. In the given figure, the center O of the circle is on the segment BD. If $AD = AC$ and $\angle D = 15^\circ$. Find the degree measure of $\angle B$.

$\angle ACD = \angle D = 15^\circ$ since $AD = AC$.

$\angle BAC = \angle ACD + \angle D = 15^\circ + 15^\circ = 30^\circ$

$\angle ACB = 90^\circ$ since AB is a diameter of the circle.

$\angle B = 90^\circ - \angle BAC = 60^\circ$



Answer: C

13. The ratio of an interior angle of a regular polygon to an exterior angle is 8 to 1. How many sides does the polygon have?

Let x be the degree measure of an interior angle of the polygon.

$$x + \frac{1}{8}x = 180^\circ$$

$$x = 160^\circ$$

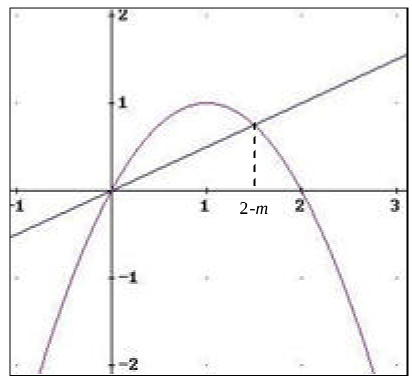
Let n be the number of the sides of the polygon.

$$\frac{n-2}{n}180^\circ = 160^\circ$$

$$n = 18$$

Answer: D

14. Find the value of m so that the line $y = mx$ divides the region enclosed by $y = 2x - x^2$ and the x-axis into two parts that have equal area.



The area of the region is

$$\int_0^2 (2x - x^2)dx = \frac{4}{3}$$

The line $y = mx$ and the curve $y = 2x - x^2$ intersect at $x = 0$ and $x = 2 - m$

After the region is divided by the line $y = mx$, the area of the top part is

$$\int_0^{2-m} ((2x - x^2) - mx)dx = \int_0^{2-m} ((2-m)x - x^2)dx = \left((2-m)\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^{2-m} = \frac{(2-m)^3}{6}$$

We have

$$\frac{(2-m)^3}{6} = \frac{2}{3}$$

$$(2-m)^3 = 4$$

$$m = 2 - \sqrt[3]{4}$$

Answer: D

15. Let $f(x) = \dfrac{3x-7}{5x-4}$. Find $f^{-1}(x)$

Solve $x = \dfrac{3y-7}{5y-4}$ for y.

$$5xy - 4x = 3y - 7 \quad \Rightarrow \quad (5x-3)y = 4x - 7 \quad \Rightarrow \quad y = \frac{4x-7}{5x-3}$$

Answer: E

16. How many subsets does $S = A \cup B$ have if $A = \{-1, 0, 1, 2\}$ and $B = \{1, 2, 5\}$?

$$S = A \cup B = \{-1, 0, 1, 2, 5\}$$

The number of all the subsets of S is

$$2^5 = 32$$

Answer: C

17. The values of y that will satisfy the system of equations

$$2x^2 + 6x + 5y + 1 = 0$$
$$2x + y + 3 = 0$$

may be found by solving which of the following:

A) $y^2 + 14y - 7 = 0$ B) $y^2 + 8y + 1 = 0$ C) $y^2 + 10y - 7 = 0$ D) $y^2 + y - 12 = 0$

Set

$$2x^2 + 6x + 5y + 1 = 0 \quad (1)$$
$$2x + y + 3 = 0 \quad (2)$$

Solve the equation (2) for x.

$$x = -\frac{y+3}{2}$$

Substitute the result into the equation (1)

$$2\left(-\frac{y+3}{2}\right)^2 + 6\left(-\frac{y+3}{2}\right) + 5y + 1 = 0$$
$$\frac{y^2 + 6y + 9}{2} - 3y - 9 + 5y + 1 = 0$$
$$y^2 + 10y - 7 = 0$$

Answer: C

18. $\cos^{-1}\left(\cos\frac{5p}{4}\right) + \sin^{-1}\left(\sin\frac{5p}{4}\right) = ?$

$$\cos^{-1}\left(\cos\frac{5p}{4}\right) + \sin^{-1}\left(\sin\frac{5p}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{3p}{4} + \left(-\frac{p}{4}\right) = \frac{p}{2}$$

Answer: A

19. A circle of radius 6 has half of its area removed by cutting away a border of uniform width. Find the width of the border.

Let x be the uniform width of the border.

$$p\,(6-x)^2 = \frac{1}{2}p\,6^2$$
$$(6-x)^2 = 18$$
$$6-x = 3\sqrt{2}$$
$$x = 6-3\sqrt{2}$$

Answer: B

20. Find the sum of the solutions to the equation $\sin x = \cos 2x$ in the interval $0 \le x \le 2p$.

$$\sin x = 1-2\sin^2 x$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$
$$\sin x = \frac{1}{2}\ ,\ x = \frac{p}{6} \text{ or } x = \frac{5p}{6};\ \sin x = -1,\ x = \frac{3p}{2}$$
$$\frac{p}{6}+\frac{5p}{6}+\frac{3p}{2} = \frac{5p}{2}$$

Answer: A

21. $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}\right) = ?$

$$\tan\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}\right) = \frac{\tan\left(\tan^{-1}\frac{1}{3}\right)+\tan\left(\tan^{-1}\frac{1}{4}\right)}{1-\tan\left(\tan^{-1}\frac{1}{3}\right)\tan\left(\tan^{-1}\frac{1}{4}\right)} = \frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\cdot\frac{1}{4}} = \frac{7}{11}$$

$$\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}\right) = \frac{\tan\left(\tan^{-1}\frac{1}{2}\right)+\tan\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}\right)}{1-\tan\left(\tan^{-1}\frac{1}{2}\right)\tan\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}\right)}$$

$$= \frac{\frac{1}{2}+\frac{7}{11}}{1-\frac{1}{2}\cdot\frac{7}{11}} = \frac{\frac{25}{22}}{\frac{15}{22}} = \frac{5}{3}$$

Answer: B

22. Find the value of $\sin 20^\circ + \sin 40^\circ - \sin 80^\circ$

$$\sin 20^\circ + \sin 40^\circ - \sin 80^\circ = 2\sin\frac{40^\circ + 20^\circ}{2}\cos\frac{40^\circ - 20^\circ}{2} - \sin 80^\circ$$

$$= 2\sin 30^\circ \cos 10^\circ - \sin 80^\circ = 2\left(\frac{1}{2}\right)\sin(90^\circ - 10^\circ) - \sin 80^\circ = 0$$

Answer: E

23.(**Tie Break No. 1**) Three circles each with a radius 8 are all tangent to each other. Find the area of the region between the circles.

Connecting the centers of the circles, we obtain an equilateral triangle with side length 16. Its area is

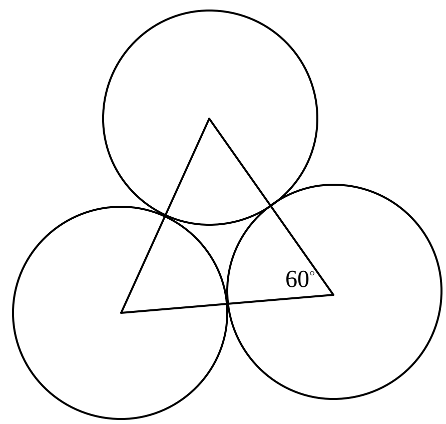
$$\frac{1}{2}(16)(16\sin 60^\circ) = 64\sqrt{3}$$

The area of the sector cut by the equilateral triangle is

$$\frac{(8^2)p}{6} = \frac{32p}{3}$$

The area of the region between the circles is

$$64\sqrt{3} - 3\left(\frac{32p}{3}\right) = 64\sqrt{3} - 32p$$



Answer: A

24. $$\frac{\sec 180^\circ \cot 45^\circ + \sin 30^\circ}{\cos 60^\circ + \cot^2 30^\circ} = ?$$

$$\frac{\sec 180^\circ \cot 45^\circ + \sin 30^\circ}{\cos 60^\circ + \cot^2 30^\circ} = \frac{(-1)\cdot 1 + \frac{1}{2}}{\frac{1}{2} + (\sqrt{3})^2} = \frac{-\frac{1}{2}}{\frac{7}{2}} = -\frac{1}{7}$$

Answer: C

25. $\left[2\left(\cos 30^\circ + i\sin 30^\circ\right)\right]^3 = ?$

By the DeMoivre's Theorem,

$$\left[2\left(\cos 30^\circ + i\sin 30^\circ\right)\right]^3 = 2^3\left(\cos 90^\circ + i\sin 90^\circ\right) = 8i$$

Answer: D

26. If $\tan\left(45^\circ + x\right) = 2$, find the value of $\cos 2x + \sin 2x$

$$2 = \tan\left(45^\circ + x\right) = \frac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} = \frac{1 + \tan x}{1 - \tan x}$$
$$2 - 2\tan x = 1 + \tan x$$
$$\tan x = \frac{1}{3}$$

We have $\cos x = \frac{3}{\sqrt{10}}$ and $\sin x = \frac{1}{\sqrt{10}}$ or $\cos x = -\frac{3}{\sqrt{10}}$ and $\sin x = -\frac{1}{\sqrt{10}}$

In the both cases, we have

$$\cos 2x = \cos^2 x - \sin^2 x = \left(\pm\frac{3}{\sqrt{10}}\right)^2 - \left(\pm\frac{1}{\sqrt{10}}\right)^2 = \frac{4}{5}$$
$$\sin 2x = 2\sin x\cos x = 2\left(\pm\frac{3}{\sqrt{10}}\right)\left(\pm\frac{1}{\sqrt{10}}\right) = \frac{3}{5}$$
$$\cos 2x + \sin 2x = \frac{7}{5}$$

Answer: C

27. How many positive 3-digit integers less than 500 can be formed using only the digits 1, 3, 5, and 7 if repetition of digits is allowed?

Since 5 and 7 can not be placed in the hundredth place, we can form $2 \times 4 \times 4 = 32$ positive 3-digit integers.

Answer: A

28.(**Tie Break No. 2**) Given that f is a function such that $f(x+y)=f(x)+f(y)+x^2y+xy^2$ for all real values of x and y; and that $\lim_{x\to 0}\frac{f(x)}{x}=1$, then $f'(x)=?$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x)+f(\Delta x)+x^2(\Delta x)+x(\Delta x)^2-f(x)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left(\frac{f(\Delta x)}{\Delta x}+x^2+x(\Delta x)\right)=1+x^2$$

Answer: D

29. In the following 3×3 matrix, select 3 different entries at random. What is the probability a random selection of 3 entries has at least two in a same row or in a same column?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The number of all the possible selection of 3 different entries is ${}_9C_3=\frac{9!}{3!\,6!}=42$

There are 6 selections of 3 entries from different rows and different columns:

$$a_{11}a_{22}a_{33},\ a_{12}a_{23}a_{31},\ a_{13}a_{21}a_{32},\ a_{13}a_{22}a_{31},\ a_{12}a_{21}a_{33},\ a_{11}a_{23}a_{32}$$

The probability a random selection has at least two in a same row or in a same column is

$$P=\frac{42-6}{42}=\frac{13}{14}$$

Answer: E

30. Find the exact value of $\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}$.

$$\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}=\sqrt{\left(\sqrt{2}\right)^2+2\sqrt{2}+1}-\sqrt{\left(\sqrt{2}\right)^2-2\sqrt{2}+1}$$
$$=\sqrt{\left(\sqrt{2}+1\right)^2}-\sqrt{\left(\sqrt{2}-1\right)^2}=\left(\sqrt{2}+1\right)-\left(\sqrt{2}-1\right)=2$$

Answer: E

31. If $r^2 + \frac{1}{r^2} = 4,$ Find the value of $\left(r + \frac{1}{r}\right)^4$

$$r^2 + \frac{1}{r^2} + 2 = 6$$

$$r^2 + 2r \cdot \frac{1}{r} + \frac{1}{r^2} = 6$$

$$\left(r + \frac{1}{r}\right)^2 = 6$$

$$\left(r + \frac{1}{r}\right)^4 = 36$$

Answer: D

32. A rectangular field is half as wide as it is long and is completely enclosed by x yards of fencing. What is the area of the rectangle in terms of x?

The width of the rectangle is $\frac{x}{6}$ and the length of the rectangle is $\frac{x}{3}$.

The area of the rectangle is $\frac{x^2}{18}$.

Answer: A

33.(**Tie Break No.3**) Solve | x – 3 | + | x + 2 | < 11.

If $x \le -2$, the inequality is equivalent to $-(x-3) - (x+2) < 11$.

$$-2x + 1 < 11$$

$$x > -5$$

$-5 < x \le -2$ is a portion of the solution set.

If $-2 < x \le 3$, the inequality is equivalent to $-(x-3) + (x+2) < 11$.

$$5 < 11 \text{ is always true.}$$

$-2 < x \le 3$ is another portion of the solution set.

If $3 < x$, the inequality is equivalent to $(x-3) + (x+2) < 11$

$$2x - 1 < 11$$

$$x < 6$$

$3 < x < 6$ is the third portion of the solution set.

The solution set of the inequality is the union of these three portions

$$-5 < x < 6.$$

Answer: C

34. Find the domain of the real function $f(x) = \ln \ln \ln \ln x$

$\ln \ln \ln x > 0$ implies $\ln \ln x > 1$
$\ln \ln x > 1$ implies $\ln x > e$
$\ln x > e$ implies $x > e^e$
The domain of the function $f(x) = \ln \ln \ln \ln x$ is $\{x \mid x > e^e\}$

Answer: D

35. In order to receive an A in a college course it is necessary to obtain an average of 90% on five one-hour exams of 100 points each and on one final exam of 250 points. If a student scores 75, 82, 90, 91, and 92 on the one-hour exams, what is the minimum score (%) on the final exam that the person can receive and still earn an A?

In order to receive an A, the student at least has to obtain total $(500 + 250) \times 0.9 = 675$ points
Therefore, the student has to obtain $675 - (75 + 82 + 90 + 91 + 92) = 245$ points on the final exam and the minimum score on the final exam is $\frac{245}{250} = 98\%$

Answer: C

36. Let $\begin{vmatrix} 3 - x & -6 \\ 5x - 6 & 8 \end{vmatrix} = 36$. Solve for x

$$36 = \begin{vmatrix} 3 - x & -6 \\ 5x - 6 & 8 \end{vmatrix} = (3 - x)8 - (-6)(5x - 6) = 22x - 12$$
$$x = \frac{24}{11}$$

Answer: B

37. Solve the equation $\left(\sqrt{5}\right)^{x+4} = 25^x$

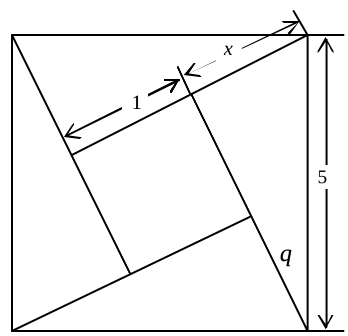
$$\left(5^{1/2}\right)^{x+4} = \left(5^2\right)^x$$
$$5^{\frac{x+4}{2}} = 5^{2x}$$
$$\frac{x+4}{2} = 2x$$
$$x = \frac{4}{3}$$

Answer: B

38. In the given figure, q is the smaller acute angle in the four congruent right triangles. The area of the larger square is 25 and the area of the smaller square is 1. Find $\sin 2q$.

Let the shortest side of the right triangle have length x.
By the Pythagorean Theorem,

$$x^2 + (x+1)^2 = 5^2$$
$$x^2 + x - 12 = 0$$
$$x = 3$$
$$\sin q = \frac{3}{5} \text{ and } \cos q = \frac{4}{5}$$
$$\sin 2q = 2\sin q \cos q = \frac{24}{25}$$



Answer: D

39. If $f(x) = x+1$ then $f(1) - f(2) + f(3) - f(4) + ... + f(99) = ?$

$$f(1) - f(2) + f(3) - f(4) + ... + f(99) = 2 - 3 + 4 - 5 + \cdots - 99 + 100$$
$$= (2 + 4 + \cdots 100) - (3 + 5 + \cdots + 99) = \frac{50(2+100)}{2} - \frac{49(3+99)}{2}$$
$$= 50(51) - 49(51) = 51$$

Answer: A

40. The future value of a particular investment is calculated by the formula $F = P\left(1+\frac{r}{n}\right)^{nt}$, where P is the principal, r is the annual rate of return, n is the number of compounds per year, and t is the number of years. Find the formula for time to triple your investment.

$$3P = P\left(1+\frac{r}{n}\right)^{nt} \Rightarrow \log 3 = \log\left(1+\frac{r}{n}\right)^{nt} \Rightarrow \log 3 = nt\log\left(1+\frac{r}{n}\right) \Rightarrow t = \frac{\log 3}{n\log\left(1+\frac{r}{n}\right)}$$

Answer: B

41.(**Tie Break No. 4**) Examine the following functions. Which of them is neither even nor odd?

I. $f(x) = \log\left(x+\sqrt{x^2+1}\right)$ II. $f(x) = \ln\frac{1-x}{1+x}$,

III. $f(x) = \frac{x^2+x^3}{1+x}$ IV. $f(x) = x\left(\frac{1}{2^{-x}-1}+\frac{1}{2}\right)$

The function I is odd because both $f(x)$ and $f(-x)$ have the same domain $\{x \mid -\infty < x < \infty\}$ and

$$f(-x) = \log\left(-x+\sqrt{(-x)^2+1}\right) = \log\frac{\left(-x+\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)}{x+\sqrt{x^2+1}}$$

$$= \log\frac{-x^2+\left(\sqrt{x^2+1}\right)^2}{x+\sqrt{x^2+1}} = \log\frac{1}{x+\sqrt{x^2+1}} = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$$

The function II is odd because both $f(x)$ and $f(-x)$ have the same domain $\{x \mid -1 < x < 1\}$ and

$$f(-x) = \ln\frac{1-(-x)}{1+(-x)} = \ln\frac{1+x}{1-x} = -\ln\frac{1-x}{1+x} = -f(x)$$

The function IV is even because both $f(x)$ and $f(-x)$ have the same domain $\{x \mid x \neq 0\}$ and

$$f(-x) = -x\left(\frac{1}{2^{-(-x)}-1}+\frac{1}{2}\right) = -x\left(\frac{2+2^x-1}{2\left(2^x-1\right)}\right) = -x\left(\frac{2^x+1}{2\left(2^x-1\right)}\cdot\frac{2^{-x}}{2^{-x}}\right)$$

$$= -x\left(\frac{1+2^{-x}}{2\left(1-2^{-x}\right)}\right) = x\left(\frac{2+\left(2^{-x}-1\right)}{2\left(2^{-x}-1\right)}\right) = x\left(\frac{1}{2^{-x}-1}+\frac{1}{2}\right) = f(x)$$

The function III is neither even nor odd because the domain of $f(x) = \frac{x^2+x^3}{1+x}$ is $\{x \mid x \neq -1\}$

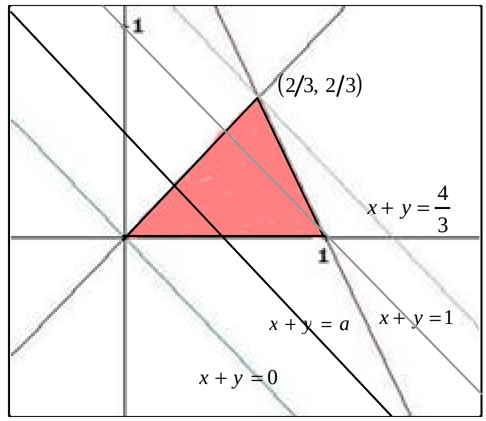
and the domain of $f(-x) = \frac{(-x)^2+(-x)^3}{1+(-x)} = \frac{x^2-x^3}{1-x}$ is $\{x \mid x \neq 1\}$

Answer: C

42. The solution set of the following system of inequalities is a triangle in the rectangular coordinate system. Find the range of the variable a.

$$\begin{cases} x - y \ge 0 \\ 2x + y \le 2 \\ y \ge 0 \\ x + y \le a \end{cases}$$

The solution set of the system $\begin{cases} x - y \ge 0 \\ 2x + y \le 2 \\ y \ge 0 \end{cases}$ is the shaded triangle in the figure.



The solution of the inequality $x + y \le a$ is the right half plane of the line $x + y = a$.

Therefore, when $0 < a \le 1$ or $a \ge \frac{4}{3}$, the solution set of the system $\begin{cases} x - y \ge 0 \\ 2x + y \le 2 \\ y \ge 0 \\ x + y \le a \end{cases}$ is a triangle.

Answer: D

43. If $f_1(x) = \frac{1}{2 - x}$, $f_2 = f_1 \circ f_1$, $f_3 = f_1 \circ f_2$, $f_4 = f_1 \circ f_3$, …., find $f_{101}(3)$

$$f_1(3) = \frac{1}{2-3} = -1,\quad f_2(3) = \frac{1}{2-(-1)} = \frac{1}{3},\quad f_3(3) = \frac{1}{2 - \frac{1}{3}} = \frac{3}{5},\quad f_4(3) = \frac{1}{2 - \frac{3}{5}} = \frac{5}{7}, \ldots$$

By math induction, we show that $f_k(3) = \frac{2k-3}{2k-1}$.

Assume $f_k(3) = \frac{2k-3}{2k-1}$ is true.

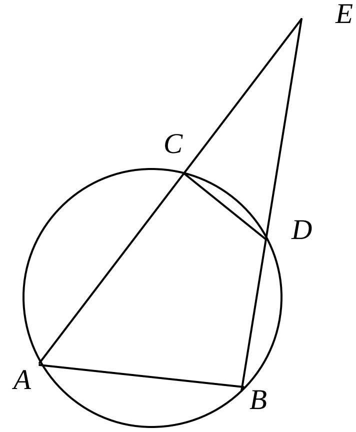
$$f_{k+1}(3) = \frac{1}{2 - \frac{2k-3}{2k-1}} = \frac{1}{\frac{(4k-2)-(2k-3)}{2k-1}} = \frac{2k-1}{2k+1} = \frac{2(k+1)-3}{2(k+1)+1}$$

The statement is true for $k+1$.

Therefore, we have $f_{101}(3) = \frac{2(101)-3}{2(101)-1} = \frac{199}{201}$

Answer: B

44. In the given figure, A, B, C, and D are four points on a circle. The segments AC and BD are extended and intersect at E. If $BD = 11$, $DE = 3$, and $AC = 19$, find the length of CE.



$\angle EDC + \angle CDB = 180^\circ$ and $\angle A + \angle CDB = 180^\circ$

Therefore, $\angle EDC = \angle A$. $\Delta EAB \sim \Delta EDC$

$$\frac{CE}{DE} = \frac{BE}{AE} = \frac{BD + DE}{AC + CE}$$

$$\frac{CE}{3} = \frac{11+3}{19 + CE}$$

$$(CE)^2 + 19(CE) - 42 = 0$$

$$((CE) - 2)((CE) + 21) = 0$$

$$CE = 2$$

Answer: E

45.(**Tie Break No. 5**) Find the minimum distance from points of $y = -x^2$ to the line $4x + 3y - 8 = 0$.

The distance from (x, y) to the line is $d = \dfrac{|4x + 3y - 8|}{\sqrt{4^2 + 3^2}}$

The distance from a point $(x, -x^2)$ of $y = -x^2$,

$$d = \frac{|4x + 3(-x^2) - 8|}{5} = \frac{\left|-\frac{20}{3} - 3\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right)\right|}{5} = \frac{\left|-\frac{20}{3} - 3\left(x - \frac{2}{3}\right)^2\right|}{5}$$

Therefore, when $x = \dfrac{2}{3}$, The distance d has the minimum value $\dfrac{\left|-\frac{20}{3}\right|}{5} = \dfrac{4}{3}$

Answer: B