1. The sum of the squares of two positive numbers is 185. If one of the numbers is 3 greater than the other, find the greater number.

Let *x* be the greater number. Then the other one is $x-3$.

$$x^2+(x-3)^2=185$$
$$2x^2-6x-176=0$$
$$2(x-11)(x+8)=0$$
$$x=11$$

Correct Answer: D

2. The base of an isosceles triangle exceeds each of the equal sides by 8 feet. If the perimeter is 89 feet, what is the length of the base in feet?

Let *x* be the length of the base.

$$x+2(x-8)=89$$
$$3x=105$$
$$x=35$$

Correct Answer: C

3. If # is a binary operation such that $a\#b$ is defined as $\left(\dfrac{a^2+b^2}{a^2-b^2}\right)$ and $(a^2-b^2\neq 0)$, then what is the value of $a\#b$ if $2a=b$ and $a\neq 0$?

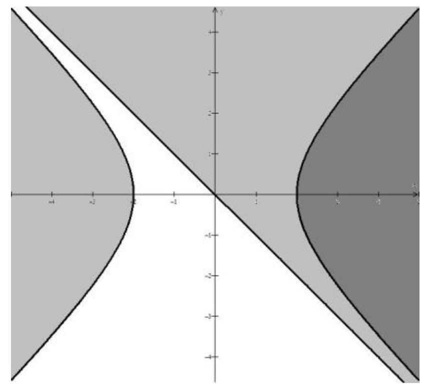
$$a\#(2a)=\left(\frac{a^2+(2a)^2}{a^2-(2a)^2}\right)=\frac{5a^2}{-3a^2}=-\frac{5}{3}$$

Correct Answer: E

4. The domain of the real-valued function $f(x,y)=\sqrt{x^2-y^2-4}+\sqrt{x+y}$ is symmetric with respect to which of the following?

The domain of $f(x,y)$ is the intersection of the regions $\{(x,y)\mid x^2-y^2\geq 4\}$ and $\{(x,y)\mid x+y\geq 0\}$

It is symmetric with respect to the *x*-axis



Correct Answer: A

5. Determine the parameter m so that the binomial $x-1$ is a factor of the polynomial $p(x)=x^3-3x^2+mx-4$.

By the Factor Theorem, $p(1)=1^3-3(1^2)+m(1)-4=0$

$m=6$

Correct Answer: B

6. In the figure, ΔABC is inscribed in a circle with center *O*. If $\angle BOC=100^\circ$ and $\angle ABO=25^\circ$, then $\angle ACO=?$

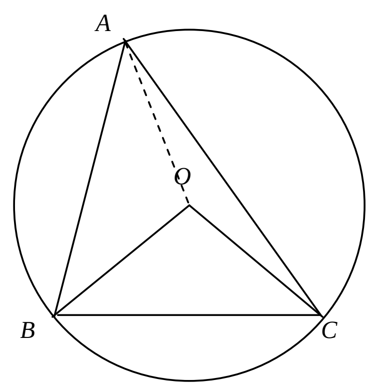
$\angle A=\frac{1}{2}\angle BOC=50^\circ$

The sum of all the angles of two triangles ΔABO and ΔACO is 360°

$\angle ACO=360^\circ-\angle ABO-(\angle BAO+\angle OAC)-(\angle AOB+\angle AOC)$

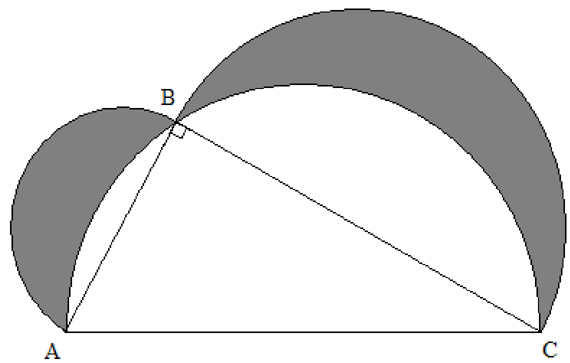
$=360^\circ-\angle ABO-\angle BAC-(360^\circ-\angle BOC)$

$=360^\circ-25^\circ-50^\circ-(360^\circ-100^\circ)=25^\circ$



Correct Answer: E

7. **(Tie Break No. 1)** In the figure, ΔABC is inscribed in a semicircle whose diameter is $AC = 17$. If two semicircles are drawn with diameters *AB* and *BC*, with $AB = 8$ and $BC = 15$, find the area of the shaded region.



The sum of the areas of the two semicircles with diameters *AB* and *BC* is

$$\frac{1}{2}\left(\left(\frac{8}{2}\right)^2 + \left(\frac{15}{2}\right)^2\right)\pi = \frac{289}{8}\pi$$

The area of the semicircle with diameter *AC* is $\frac{1}{2}\left(\frac{17}{2}\right)^2 \pi = \frac{289}{8}\pi$

The area of ΔABC is $\frac{1}{2}(8)(15) = 60$

The area of the shaded region is $\frac{289}{8}\pi - \left(\frac{289}{8}\pi - 60\right) = 60$

Correct Answer: B

8. If $f(n) = \begin{cases} n^2 & n \text{ is odd} \\ -n^2 & n \text{ is even} \end{cases}$ and $a_n = f(n) + f(n+1)$, find the sum $S_{100} = \sum_{n=1}^{100} a_n$.

$a_{2k-1} = (2k-1)^2 - (2k)^2 = -4k+1$, $a_{2k} = -(2k)^2 + (2k+1)^2 = 4k+1$

Then, $a_{2k-1} + a_{2k} = 2$

$$S_{100} = \sum_{n=1}^{100} a_n = \sum_{k=1}^{50} (a_{2k-1} + a_{2k}) = \sum_{k=1}^{50} 2 = 50(2) = 100$$

Correct Answer: A

9. A circle passes though the points $(0, 0)$ and $(2, 4)$. Its center is on the line $2x - y = 5$. Find the equation of the circle.

Let the center of the circle be (h, k).

The points $(0, 0)$ and $(2, 4)$ being on the circle, $(h-0)^2 + (k-0)^2 = (h-2)^2 + (k-4)^2$, $4h + 8k = 20$. Since (h, k) is on the line $2x - y = 5$, we have $2h - k = 5$

Solving the system $\begin{cases} h + 2k = 5 \\ 2h - k = 5 \end{cases}$, we have $(h, k) = (3, 1)$

The equation of the circle is $(x-3)^2 + (y-1)^2 = (3-0)^2 + (1-0)^2$

That is $x^2 + y^2 - 6x - 2y = 0$

Correct Answer: B

10. Marbles are to be drawn at random without replacement from a bag containing only black and white marbles. If two marbles are randomly drawn, the probability that they are both white is $1/3$. If, however, three marbles are randomly drawn from this bag, the probability that all three are white is $1/6$. How many marbles are in the bag?

Let n be the number of marbles in the bag and k the number of white marbles in the bag. We have

$$\frac{{}_kC_2}{{}_nC_2}=\frac{1}{3}\,,\ \frac{k(k-1)}{n(n-1)}=\frac{1}{3}\quad (1)\ \text{ and }\ \frac{{}_kC_3}{{}_nC_3}=\frac{1}{6}\,,\ \frac{k(k-1)(k-2)}{n(n-1)(n-2)}=\frac{1}{6}\quad (2).$$

Dividing (2) by (1), we have $\frac{k-2}{n-2}=\frac{1}{2}$ and $k=\frac{n+2}{2}$ (3)

Substituting (3) into (1), $3\left(\frac{n+2}{2}\right)\left(\frac{n}{2}\right)=n(n-1)$, Since $n\neq 0$, $3(n+2)=4(n-1)$ and $n=10$

Correct Answer: B

11. $\frac{2x^3-23x^2+85x-95}{(x-4)^4}$ is equal to which of the following expressions?

By synthetic division,

$$\begin{array}{r|rrrr} 4 & 2 & -23 & 85 & -95 \\ & 0 & 8 & -60 & 100 \\ \hline & 2 & -15 & 25 & |5 \end{array}\ ,\quad \begin{array}{r|rrr} 4 & 2 & -15 & 25 \\ & 0 & 8 & -28 \\ \hline & 2 & -7 & |-3 \end{array}\ ,\quad \text{and}\quad \begin{array}{r|rr} 4 & 2 & -7 \\ & 0 & 8 \\ \hline & 2 & |1 \end{array}$$

$$\frac{2x^3-23x^2+85x-95}{(x-4)^4}=\frac{2x^2-15x+25+\frac{5}{x-4}}{(x-4)^3}=\frac{2x-7-\frac{3}{x-4}+\frac{5}{(x-4)^2}}{(x-4)^2}$$

$$=\frac{2+\frac{1}{x-4}-\frac{3}{(x-4)^2}+\frac{5}{(x-4)^3}}{x-4}=\frac{2}{(x-4)}+\frac{1}{(x-4)^2}-\frac{3}{(x-4)^3}+\frac{5}{(x-4)^4}$$

Correct Answer: B

12. If m is an integer such that $-5<m<2$, and n is an integer such that $-4<n<5$, what is the least possible value for $3m^2-2n$?

$3m^2$ has minimum value 0 for $-5<m<2$ and $2n$ has maximum value 8 for $-4<n<5$. The least possible value for $3m^2-2n$ is $0-8=-8$.

Correct Answer: D

13. The number of bacteria in a certain culture is related to time by $N = N_0 e^{-0.5t}$, where N_0 is the amount present initially and t is elapsed time in hours. If the initial number of bacteria was 500,000, how much time would elapse for the number of bacteria to be 100,000?

$100{,}000 = 500{,}000 e^{-0.5t}$, $0.2 = e^{-0.5t}$, $\ln 0.2 = \ln e^{-0.5t}$, $\ln 0.2 = -0.5t$, and

$t = -2\ln 0.2 = -2\ln \frac{1}{5} = 2\ln 5$

Correct Answer: D

14. Find the solution set for the system $\begin{cases} \frac{2}{x} + \frac{4}{y} = 1 \\ y - 4x = 0 \end{cases}$

Replace y in the first equation with $4x$

$\frac{2}{x} + \frac{4}{4x} = 1$, $\frac{3}{x} = 1$, and $x = 3$. Then $y = 12$

Correct Answer: B

15. If $\begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 8 & x & 6 \end{vmatrix} = -8$, solve for x.

$$-8 = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 8 & x & 6 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ x & 6 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -2 \\ 8 & 6 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 8 & x \end{vmatrix}$$

$(6 + 2x) + 2(24 + 16) + 3(4x - 8) = -8$

$14x + 62 = -8$

$14x = -70$ and $x = -5$

Correct Answer: A

16. For a triangle, the differences between the semi-perimeter and the sides are 3, 4, and 5 respectively. What is the area of the triangle?

Let *a*, *b*, and *c* be the lengths of three sides of the triangle and the semi-perimeter

$s=\frac{1}{2}(a+b+c)=(s-a)+(s-b)+(s-c)=3+4+5=12$

By Heron's formula, the area of the triangle is

$\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{12(3)(4)(5)}=12\sqrt{5}$

Correct Answer: C

17. **(Tie Break No. 2)** In the figure, the length of circle's radius is 3 and its center is at *O*. *AC* is tangent to the circle at *C*. $AC=2\sqrt{3}$ and $AB=6$. Find the distance from *O* to *AB*.

$OC\perp AC$. By the Pythagorean Theorem, $(AO)^2=(AC)^2+(OC)^2=\left(2\sqrt{3}\right)^2+3^2=21$

Let $OD\perp AB$ and $BD=x$. Then $AD=6-x$

By the Pythagorean Theorem,

$(BO)^2-(BD)^2=(OD)^2=(AO)^2-(AD)^2$

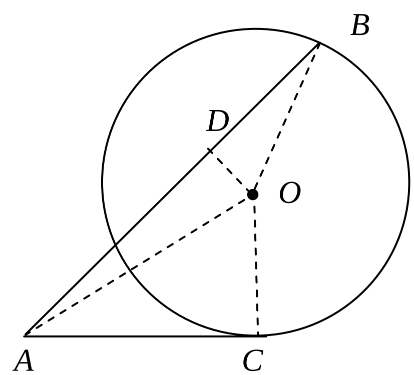
We have

$3^2-x^2=21-(6-x)^2$

$9-x^2=21-36+12x-x^2$

$BD=x=2$

$OD=\sqrt{(OB)^2-(BD)^2}=\sqrt{3^2-2^2}=\sqrt{5}$



Correct Answer: E

18. Find the domain of the function $f(x)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}$

$1+\frac{1}{x}=0$ implies $x=-1$

$1+\cfrac{1}{1+\cfrac{1}{x}}=0$ implies $1+\frac{1}{x}=-1$ and $x=-\frac{1}{2}$

The domain of $f(x)$ is $\left\{x \mid x\neq -1,\ x\neq -\frac{1}{2},\ x\neq 0\right\}$

Correct Answer: C

19. $\sin\left(\frac{7\pi}{12}\right) = ?$

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Correct Answer: C

20. Jane is 6 years older than Tom, and Tom is 5 years younger than Phillip. Chris is 3 years older than Tom. If Jane's age is expressed as *J*, what is the sum of the ages of Jane, Tom, Phillip, and Chris in terms of *J*?

The age of Tom is $J-6$, the age of Phillip $(J-6)+5$, and the age of Chris $(J-6)+3$
The sum of their ages is $J+(J-6)+((J-6)+5)+((J-6)+3) = 4J-10$

Correct Answer: A

21. Solve the inequality $0.2\left(x^2-5x+4\right)\left(x^2-24x+144\right)\left(x-20\right) \ge 0$

$p(x) = 0.2\left(x^2-5x+4\right)\left(x^2-24x+144\right)\left(x-20\right) = 0.2(x-1)(x-4)(x-12)^2(x-20)$
The number line is divided by the zeros 1, 4, 12, and 20 of $p(x)$ into 5 intervals.
$P(0)<0$, $p(2)>0$, $p(5)<0$, $p(12)=0$, $p(13)<0$, and $p(21)>0$
Therefore, the solution set is $[1,4]\cup\{12\}\cup[20,\infty)$ by the continuity of polynomial function.

Correct Answer: D

22. Suppose $f(x)$ is a function such that $f(x) = f^{-1}(x)$, and $f(0) = -\frac{1}{3}$. Evaluate

$$f^{-1}\left(\frac{1}{3} + f\left(\frac{1}{3} + f^{-1}\left(\frac{1}{3} + f(0)\right)\right)\right)$$

$$f^{-1}(0) = f(0) = -\frac{1}{3}$$

$$f^{-1}\left(\frac{1}{3} + f\left(\frac{1}{3} + f^{-1}\left(\frac{1}{3} + f(0)\right)\right)\right) = f^{-1}\left(\frac{1}{3} + f\left(\frac{1}{3} + f^{-1}\left(\frac{1}{3} + \left(-\frac{1}{3}\right)\right)\right)\right) = f^{-1}\left(\frac{1}{3} + f\left(\frac{1}{3} + f^{-1}(0)\right)\right)$$

$$f^{-1}\left(\frac{1}{3} + f\left(\frac{1}{3} + \left(-\frac{1}{3}\right)\right)\right) = f^{-1}\left(\frac{1}{3} + f(0)\right) = f^{-1}\left(\frac{1}{3} + \left(-\frac{1}{3}\right)\right) = f^{-1}(0) = f(0) = -\frac{1}{3}$$

Correct Answer: D

23. Multiply and simplify $\sqrt[3]{3} \cdot \sqrt{10}$

$$\sqrt[3]{3} \cdot \sqrt{10} = 3^{\frac{1}{3}} \cdot 10^{\frac{1}{2}} = \left(3^2\right)^{\frac{1}{6}} \cdot \left(10^3\right)^{\frac{1}{6}} = (9 \cdot 1000)^{\frac{1}{6}} = \sqrt[6]{9000}$$

Correct Answer: C

24. **(Tie Break No. 3)** In a trapezoid *ABCD*, $AD // BC$. The diagonals *AC* and *BD* intersect at point *O*. The area of ΔAOD is 5 and the area ΔBOC is 9. Find the area of the trapezoid.

Let *EF* pass through the point *O* and be perpendicular to *AD* and *BC*.

The area of $\Delta AOD = \frac{1}{2}(AD)(EO) = 5$

The area of $\Delta BOC = \frac{1}{2}(BC)(OF) = 9$

$\Delta AOD \sim \Delta BOC$

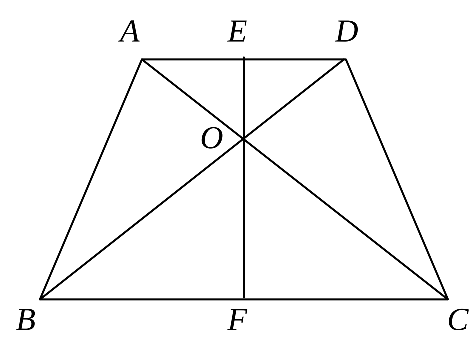
$\frac{EO}{OF} = \frac{AD}{BC} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

The area of the trapezoid is equal to

$\frac{1}{2}(AD + BC)(EO + OF) =$

$= \frac{1}{2}(AD)\left(EO + \frac{3}{\sqrt{5}}EO\right) + \frac{1}{2}(BC)\left(\frac{\sqrt{5}}{3}OF + OF\right)$

$= 5 + \frac{3}{\sqrt{5}} \cdot 5 + \frac{\sqrt{5}}{3} \cdot 9 + 9 = 14 + 6\sqrt{5}$



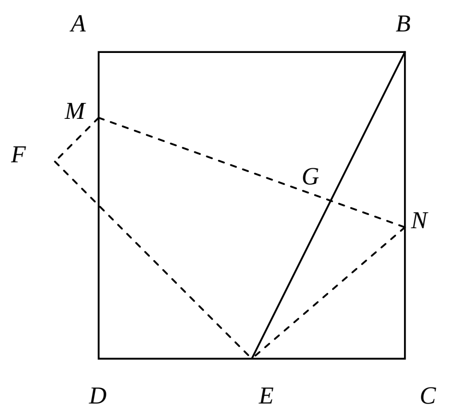
Correct Answer: E

25. Determine the constant term in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$

The general term of the expansion is ${}_{12}C_j(x^2)^{12-j}(x^{-1})^j$.
$2(12-j)-j=0$ when $j=8$
The constant term is ${}_{12}C_8=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}=495$

Correct Answer: C

26. In the figure, the sides of rectangle *ABCD* have length 8. If it is folded along the dashed line *MN*, the vertex *B* is positioned at the mid-point E of the side *CD* and the vertex *A* is positioned at point *F*. Find the length of *CN*.

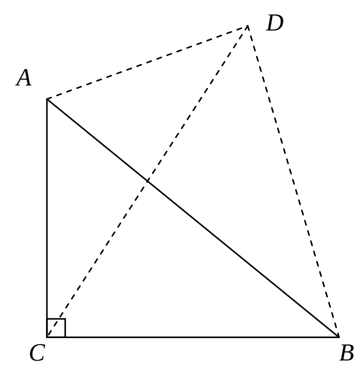


Connect *BE* and mark the intersection point *G* of *BE* and *MN*. $\Delta NBG \cong \Delta NEG$ and $BE \perp MN$. Let $CN=x$. Then $BN=8-x$. By Pythagorean Theorem,
$(EC)^2+(CN)^2=(EN)^2=(BN)^2$
We have $4^2+x^2=(8-x)^2$ and $x=CN=3$

Correct Answer: C

27. In ΔABC, $\angle ACB=90^\circ$, $AC=3$ and $BC=4$. *D* is a point outside of the triangle. If $AD=\sqrt{10}$ and $CD=5$, which of the following may be the length of BD.

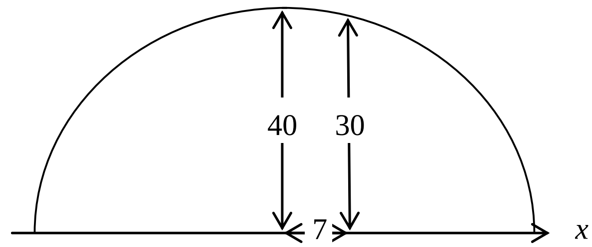
Set the triangle in a rectangular coordinate system. Place *C* at the origin, *CB* in the positive direction of the *x*-axis, and *CA* in the positive direction of the *y*-axis. Let (x, y) be the coordinates of *D*.



$(AD)^2=x^2+(y-3)^2=10$
$(CD)^2=x^2+y^2=25$
Solve the system
$(x,\ y)=(\pm 3, 4)$
$BD=\sqrt{(3-4)^2+(4-0)^2}=\sqrt{17}$ or
$BD=\sqrt{(-3-4)^2+(4-0)^2}=\sqrt{65}$

Correct Answer: C

28. A railroad tunnel is shaped like a semi-ellipse. The height of the tunnel at the center is 40 *ft* and the vertical clearance must be 30 *ft* at a point 7 *ft* from the center. Find an equation for the ellipse



Let $\frac{x^2}{a^2}+\frac{y^2}{40^2}=1$ be the equation of the ellipse.

We have $\frac{7^2}{a^2}+\frac{30^2}{40^2}=1$

$a^2=112$

The equation of the ellipse is $\frac{x^2}{112}+\frac{y^2}{1600}=1$

Correct Answer: B

29. Solve the inequality $|3x+9|<|x-1|$

Solve the equation $|3x+9|=|x-1|$

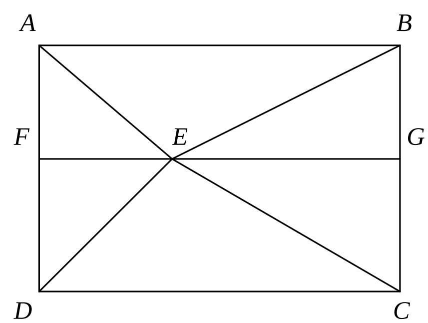
$3x+9=x-1$, $x=-5$ or $3x+9=-x+1$, $x=-2$. -5 and -2 divide the number into three intervals. In each interval pick a point to check the inequality.

$|3(-6)+9|<|(-6)-1|$ (False), $|3(-3)+9|<|(-3)-1|$ (True), and $|3(0)+9|<|(0)-1|$ (False)

The solution set is $(-5,-2)$

Correct Answer: C

30. In the rectangle *ABCD*, ΔAED and ΔBEC are both isosceles with $AE=DE$ and $BE=CE$. If the ratio of the area of ΔAED to the area of ΔBEC is 2:3, find the ratio of the area of ΔAED to the area of ΔAEB.



Draw *FG* through *E* and parallel to *AB*

We have $\frac{EG}{FE}=\frac{3}{2}$

The area of ΔAED is $\frac{1}{2}(AD)(FE)$

The area of ΔAEB is

$\frac{1}{2}(AF)(AB)=\frac{1}{2}\left(\frac{1}{2}(AD)\right)((FE)+(EG))$

$=\frac{1}{2}\left(\frac{1}{2}(AD)\right)\left((FE)+\frac{3}{2}(FE)\right)=\frac{1}{2}\left(\frac{5}{4}(AD)(FE)\right)$

The ratio of the area of ΔAED to the area of ΔAEB is $\frac{1}{\frac{5}{4}}=\frac{4}{5}$

Correct Answer: C

31. Solve $\tan x = 2\sin x$ giving all solutions between $-\pi$ and π, inclusive.

$$\tan x - 2\sin x = 0,\ \frac{\sin x}{\cos x} - 2\sin x = 0,\ \sin x\left(\frac{1}{\cos x} - 2\right) = 0$$

$$\sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$x = -\pi, 0, \pi \text{ or } x = \frac{\pi}{3}, -\frac{\pi}{3}$$

Correct Answer: E

32. If $\tan\alpha = 2, \tan\beta = 3$, and $\theta = \frac{\alpha+\beta}{2}$, find the value of $\frac{\sin\theta\cos\theta}{\sin\left(\theta - \frac{\pi}{4}\right)\sin\left(\theta + \frac{\pi}{4}\right)}$

$$\frac{\sin\theta\cos\theta}{\sin\left(\theta - \frac{\pi}{4}\right)\sin\left(\theta + \frac{\pi}{4}\right)} = \frac{\frac{2\sin\theta\cos\theta}{2}}{\frac{\cos\left(\left(\theta + \frac{\pi}{4}\right) - \left(\theta - \frac{\pi}{4}\right)\right) - \cos\left(\left(\theta + \frac{\pi}{4}\right) + \left(\theta - \frac{\pi}{4}\right)\right)}{2}}$$

$$= \frac{\sin 2\theta}{\cos\frac{\pi}{2} - \cos 2\theta} = -\frac{\sin 2\theta}{\cos 2\theta} = -\tan 2\theta = -\tan(\alpha+\beta) = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\frac{2+3}{1 - 2\cdot 3} = 1$$

Correct Answer: D

33. A sequence $\{a_n\}$ is defined by $a_0 = 7$, $a_1 = 3$, and $3a_n = 2a_{n-1} + a_{n-2}$ $(n \geq 2)$. Find $\lim_{n\to\infty} a_n$.

$$a_2 = \frac{2}{3}(3) + \frac{1}{3}(7) = \frac{13}{3} = 4 + \frac{1}{3},\quad a_3 = \frac{2}{3}\left(\frac{13}{3}\right) + \frac{1}{3}(3) = \frac{35}{9} = 4 - \frac{1}{9},$$

$$a_4 = \frac{2}{3}\left(\frac{35}{9}\right) + \frac{1}{3}\left(\frac{13}{3}\right) = \frac{109}{27} = 4 + \frac{1}{27}, \cdots$$

Assume that $a_{n-1} = 4 - \left(-\frac{1}{3}\right)^{n-2}$ and $a_n = 4 - \left(-\frac{1}{3}\right)^{n-1}$

$$a_{n+1} = \frac{2}{3}\left(4 - \left(-\frac{1}{3}\right)^{n-1}\right) + \frac{1}{3}\left(4 - \left(-\frac{1}{3}\right)^{n-2}\right) = 4 - \left(-\frac{1}{3}\right)^{n-1}\left(\frac{2}{3} + \frac{1}{3}\left(-\frac{3}{1}\right)\right) = 4 - \left(-\frac{1}{3}\right)^{n}$$

By math induction, $a_n = 4 - \left(-\frac{1}{3}\right)^{n-1}$. Therefore, $\lim_{n\to\infty} a_n = 4$.

Correct Answer: D

34. The surface area of a large spherical balloon is doubled. By what factor is the volume increased?

Let r be the radius of the sphere before and r_1 the radius after its surface area is doubled.
$4\pi r_1^2 = 2(4\pi r^2)$, $r_1 = r\sqrt{2}$
The volume of the sphere after its surface area is doubled is
$V_1 = \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi (r\sqrt{2})^3 = 2\sqrt{2}\left(\frac{4}{3}\pi r^3\right)$

Correct Answer: B

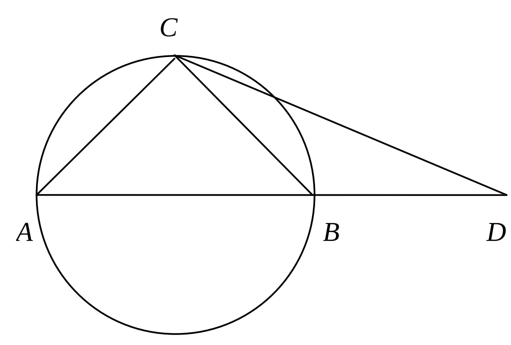
35. In a 110-volt circuit having a resistance of 11 ohms, the power, W (Watts) is given by $W = 110I - 11I^2$ when a current, I (Amps), is flowing. Determine the maximum power that can be delivered in this circuit?

$W = 110I - 11I^2 = 275 - (275 - 110I + 11I^2) = 275 - 11(5 - I)^2$
The maximum value of W is 275.

Correct Answer: C

36. Triangles ΔABC and ΔCBD are each isosceles with $AC = BC = BD$. If ΔABC is inscribed in a circle with radius 1 and AB is a diameter of the circle, find CD.

$\angle ACB = 90^\circ$, $\angle CBA = 45^\circ$, and $\angle CBD = 135^\circ$
$BD = BC = (AB)\sin 45^\circ = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$
By the Law of Cosines,
$CD = \sqrt{(BC)^2 + (BD)^2 - 2(BC)(BD)\cos(\angle CBD)}$
$= \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 - 2(\sqrt{2})(\sqrt{2})\cos 135^\circ}$
$= \sqrt{4 + 2\sqrt{2}}$



Correct Answer: D

37. If the angle θ is terminated in the first quadrant and $\cos\theta = \frac{3}{5}$, find the value of $\tan\theta$.

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{1}{\cos\theta}\right)^2 - 1} = \sqrt{\left(\frac{5}{3}\right)^2 - 1} = \frac{4}{3}$$

Correct Answer: E

38. If $f(x) = 2a\sin x\cos x + 2b\cos^2 x$, $f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + 6$, and $f(0) = 8$, find the maximum value of $f(x)$.

$$f(0) = 2a\sin 0\cos 0 + 2b\cos^2 0 = 2b = 8,\ b = 4$$
$$f\left(\frac{\pi}{6}\right) = 2a\sin\frac{\pi}{6}\cos\frac{\pi}{6} + 2b\cos^2\frac{\pi}{6} = \frac{\sqrt{3}}{2}a + \frac{3}{2}b = \frac{3\sqrt{3}}{2} + 6,\ a = 3$$
$$f(x) = 6\sin x\cos x + 8\cos^2 x = 3(2\sin x\cos x) + 4(2\cos^2 x - 1) + 4$$
$$= 3\sin 2x + 4\cos 2x + 4 = 5\left(\frac{3}{5}\sin 2x + \frac{4}{5}\cos 2x\right) + 4 = 5\sin\left(2x + \tan^{-1}\left(\frac{4}{3}\right)\right) + 4$$

Therefore, the maximum value of $f(x)$ is $5 + 4 = 9$

Correct Answer: C

39. **(Tie Break No. 4)** If $\tan\frac{\alpha+\beta}{2} = \frac{\sqrt{6}}{2}$ and $\tan\alpha\tan\beta = \frac{13}{7}$, find the value of $\cos(\alpha - \beta)$.

$$\cos 2C = \cos^2 C - \sin^2 C = \frac{\cos^2 C - \sin^2 C}{\cos^2 C + \sin^2 C} = \frac{1 - \tan^2 C}{1 + \tan^2 C}$$
$$\cos(\alpha+\beta). = \frac{1 - \tan^2\frac{\alpha+\beta}{2}}{1 + \tan^2\frac{\alpha+\beta}{2}} = \frac{1 - \left(\frac{\sqrt{6}}{2}\right)^2}{1 + \left(\frac{\sqrt{6}}{2}\right)^2} = \frac{1 - \frac{6}{4}}{1 + \frac{6}{4}} = \frac{-2}{10} = -\frac{1}{5},$$
$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = -\frac{1}{5} \quad (1)$$

$\tan\alpha\tan\beta = \frac{13}{7}$ implies $7\sin\alpha\sin\beta = 13\cos\alpha\cos\beta$ (2)

Combining (1) and (2), we have $\sin\alpha\sin\beta = \frac{13}{30}$ and $\cos\alpha\cos\beta = \frac{7}{30}$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{7}{30} + \frac{13}{30} = \frac{2}{3}$$

Correct Answer: B

40. Find the value of $\sin^2 20^\circ + \cos^2 50^\circ + \sin 20^\circ \cos 50^\circ$

$$\begin{aligned}
&\sin^2 20^\circ + \cos^2 50^\circ + \sin 20^\circ \cos 50^\circ \\
&= \frac{1-\cos 40^\circ}{2} + \frac{1+\cos 100^\circ}{2} + \frac{1}{2}\left(\sin(50^\circ + 20^\circ) - \sin(50^\circ - 20^\circ)\right) \\
&= 1 + \frac{1}{2}\left(\cos 100^\circ - \cos 40^\circ\right) + \frac{1}{2}\sin 70^\circ - \frac{1}{2}\sin 30^\circ \\
&= 1 + \frac{1}{2}\left(-2\sin\frac{100^\circ + 40^\circ}{2}\sin\frac{100^\circ - 40^\circ}{2}\right) + \frac{1}{2}\sin 70^\circ - \frac{1}{2}\sin 30^\circ \\
&= 1 - \sin 70^\circ \sin 30^\circ + \frac{1}{2}\sin 70^\circ - \frac{1}{2}\sin 30^\circ \\
&= 1 - \frac{1}{2}\sin 30^\circ = \frac{3}{4}
\end{aligned}$$

Correct Answer: A

41. Find the exact value of $\tan\frac{\pi}{8}$

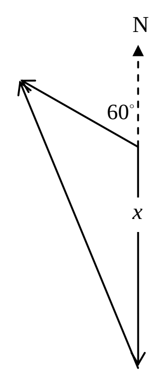
$$\tan\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1$$

Correct Answer: D

42. An airplane leaves an aircraft carrier and flies due south at 400 km/hr. The carrier is heading in the direction North 60° West of at 32 km/hr. If the plane has 5 hours of fuel, what is the maximum distance south the plane can travel so that the fuel remaining will allow a safe return to the carrier at 400 km/hr?

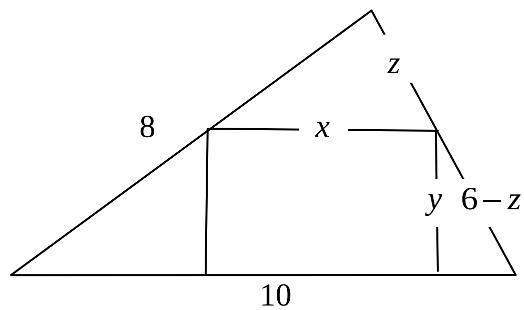
Let x be the maximum distance south the plane can travel.
By the Law of Cosines,

$$\begin{aligned}
&x^2 + (32(5))^2 - 2x(32(5))\cos 120^\circ = (400(5) - x)^2 \\
&x^2 + 25600 + 160x = 4000000 - 4000x + x^2 \\
&4160x = 3974400 \\
&x \approx 955
\end{aligned}$$



Correct Answer: A

43. A rectangle is inscribed in a right triangle as shown in the accompanying figure. If the three sides of the triangle have the lengths 6, 8, and 10 respectively, find the maximum area of the rectangle.



By the property of similar triangles, we have

$\frac{z}{x} = \frac{6}{10}$ and $\frac{y}{6-z} = \frac{8}{10}$

$z = \frac{3}{5}x$ and $y = \frac{4}{5}(6-z)$

$y = \frac{4}{5}\left(6 - \frac{3}{5}x\right) = \frac{24}{5} - \frac{12}{25}x$

Area of the rectangle is

$A(x) = xy = \frac{24}{5}x - \frac{12}{25}x^2 = 12 - 12\left(1 - \frac{1}{5}x\right)^2$

The maximum area of the rectangle is 12.

Correct Answer: B

44. Find $A + B$, where A and B are integers satisfying the expression $\sin 3\theta = A\sin\theta + B\sin^3\theta$

$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$A = 3,\ B = -4,$ and $A + B = -1$

Correct Answer: A

45. **(Tie Break No. 5)** Ali and Bill play a game in which they alternately toss a pair of fair, six-sided dice. The first one to get a total of 7 on a single toss of the dice wins the game. Find the probability that the player who tosses first will win the game.

On any single toss of dice the probability of winning is $1/6$ and of loss $5/6$. Since "first" player only tosses every other turn, the probability of this person winning the game is the probability of winning on Toss 1 $\cup$ Toss 3 $\cup$ Toss 5 $\cup \cdots$. It is equal to

$$\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \cdots = \frac{1}{6} + \left(\frac{25}{36}\right)\frac{1}{6} + \left(\frac{25}{36}\right)^2\frac{1}{6} + \cdots = \frac{1}{6}\left(\frac{1}{1 - \frac{25}{36}}\right) = \frac{6}{11}$$

Correct Answer: D