1. If $3x^2 - 2x + 7 = 0$, then $\left(x - \frac{1}{3}\right)^2 = ?$

A. $20/9$ B. $7/9$ C. $-7/9$ D. $-20/9$ E. $-8/9$

$x^2 - \frac{2}{3}x + \frac{7}{3} = 0$, $x^2 - \frac{2}{3}x = -\frac{7}{3}$, $x^2 - \frac{2}{3}x + \frac{1}{9} = -\frac{7}{3} + \frac{1}{9}$, and $\left(x - \frac{1}{3}\right)^2 = -\frac{20}{9}$

Correct Answer: D

2. Solve the equation $e^{-x^2} = \left(e^x\right)^2 \cdot \frac{1}{e^3}$

A. $\{-3\}$ B. $\{3, 1\}$ C. $\{-3, 1\}$ D. $\{1\}$ E. ϕ

$e^{-x^2} = \frac{e^{2x}}{e^3}$, $e^{-x^2} = e^{2x-3}$, $-x^2 = 2x - 3$, $x^2 + 2x - 3 = 0$, and $(x+3)(x-1) = 0$.
Then $x = -3$ or $x = 1$.

Correct Answer: C

3. A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point where the receiver is located. If the dish is 8 feet across at its opening and 3 feet deep at its center, at what position should the receiver be placed? That is, where is the focus?

A. $1\frac{1}{3}$ feet above the vertex along its axis of symmetry
B. 3 feet above the vertex along its axis of symmetry
C. At the vertex
D. $\frac{4}{3}$ inches above the vertex along its axis of symmetry
E. Not enough information to determine a solution

Let *y*-axis be the axis of symmetry of the parabola and the origin at its vertex. The equation of the parabola has the form $4py = x^2$. It passes through the points $(-4, 3)$ and $(4, 3)$, $4p(3) = (\pm 4)^2$, $p = \frac{16}{12} = \frac{4}{3}$. The focus is $\frac{4}{3}$ feet above the vertex along its axis of symmetry.

Correct Answer: A

4. Find the coefficient of x^2 in the expansion of $\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)^8$

A. 9 B. 252 C. 28 D. 84 E. 756

The general term of the expansion is $\binom{8}{r}\left(\sqrt{x}\right)^{8-r}\left(\frac{3}{\sqrt{x}}\right)^r$. We have $\frac{1}{2}(8-r)-\frac{1}{2}r=2$, $r=2$, The coefficient of x^2 is $\binom{8}{2}\cdot(3)^2=\frac{8\cdot 7}{2\cdot 1}\cdot 9=252$.

Correct Answer: B

5. In a quadrilateral *ABCD*, $\angle ABC=70^\circ$, $\angle C=90^\circ$, $AB=DB=10$, and $CD=5$. Find $\angle A$.

A. 60° B. 30° C. 20° D. 70° E. 65°

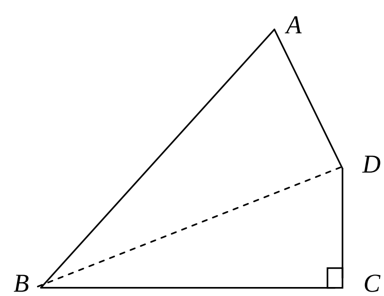
In the right ΔBCD, $CD=\frac{1}{2}DB$ implies

$\angle DBC=30^\circ$.

$\angle ABD=70^\circ-30^\circ=40^\circ$

Since $AB=DB$, $\angle A=\angle BDA=\frac{1}{2}\left(180^\circ-\angle ABD\right)$.

$\angle A=\frac{1}{2}\left(180^\circ-40^\circ\right)=70^\circ$



Correct Answer: D

6. Solve for *x*: $\begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix}=-4x$

A. $\{0,-1/2\}$ B. $\{-1/2\}$ C. $\{0\}$ D. $\{0,1/2\}$ E. $\{1\}$

$$\begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix}=x\cdot\begin{vmatrix} x & 3 \\ 1 & 2 \end{vmatrix}-1\cdot\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}+0\cdot\begin{vmatrix} 1 & 2 \\ x & 3 \end{vmatrix}=2x^2-3x,\ 2x^2-3x=-4x,\ 2x^2+x=0$$

$x(2x+1)=0$, $x=0$ or $x=-\frac{1}{2}$.

Correct Answer: A

7. What is the length of the edge of a cube if its volume could be doubled by an increase of 6 centimeters in one edge, an increase of 12 centimeters in a second edge, and a decrease of 4 centimeters in the third edge?

A. 6 *cm* B. 36 *cm* C. 6 or 12 *cm* D. 144 *cm* E. $\sqrt[3]{6}$ *cm*

Let x be the length of edge of the cube.

$(x+6)(x+12)(x-4)=2x^3$

$x^3+14x^2-288=2x^3$, $x^3-14x^2+288=0$

By the Rational Zero Test, $x=6$ is a solution of the equation. We have $(x-6)(x^2-8x-48)=0$, $(x-6)(x-12)(x+4)=0$. The length of edge of the cube can be 6 or 12.

Correct Answer: C

8. (Tie Break No. 1) $\cos^3 20^\circ+\cos^3 140^\circ+\cos^3 100^\circ=?$

A. $3/4$ B. $-3/8$ C. $3/8$ D. $-\sqrt{3}/8$ E. $-3/4$

We have $\cos 3A=4\cos^3 A-3\cos A$ and $\cos^3 A=\frac{1}{4}(\cos 3A+3\cos A)$.

$$\cos^3 20^\circ+\cos^3 140^\circ+\cos^3 100^\circ$$
$$=\frac{1}{4}(\cos 60^\circ+\cos 420^\circ+\cos 300^\circ)+\frac{3}{4}(\cos 20^\circ+\cos 140^\circ+\cos 100^\circ)$$
$$=\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)+\frac{3}{4}\left(\cos 20^\circ+2\cos\frac{140^\circ+100^\circ}{2}\cos\frac{140^\circ-100^\circ}{2}\right)$$
$$=\frac{3}{8}+\frac{3}{4}(\cos 20^\circ+2\cos 120^\circ\cos 20^\circ)$$
$$=\frac{3}{8}+\frac{3}{4}\left(\cos 20^\circ+2\left(-\frac{1}{2}\right)\cos 20^\circ\right)=\frac{3}{8}$$

Correct Answer: C

9. Solve the equation $\frac{b+c}{x+a}=\frac{b-c}{x-a}$, where $a\neq 0$, b, and $c\neq 0$ are constants.

A. $x=\frac{ab}{c}$ B. $x=ab$ C. $x=\frac{bc}{a}$ D. $x=ab-c$ E. $x=\frac{ac}{b}$

$(b+c)(x-a)=(b-c)(x+a)$, $bx-ab+cx-ac==bx+ab-cx-ac$, $2cx=2ab$.

$x=\frac{ab}{c}$

Correct Answer: A

10. Reversing the digits of Alice's age gives her mother's age. The difference in their ages is 27 years. If the difference of the digits in each age is 3, what is the sum of their ages?

A. 76 B. 132 C. 87 D. 67 E. 143

Let x and y be digital numbers and $10x+y$, the age of Alice. Then we have the system

$$\begin{cases}(10y+x)-(10x+y)=27 & (1)\\ y-x=3 & (2)\end{cases}.$$ Replace y in (1) with $(x+3)$.

We have $(10(x+3)+x)-(10x+(x+3))=27$ and $27=27$. Therefore, x can be any digital number less than 7. The sum of their ages is $(10y+x)+(10x+y)=11(x+y)$. It must be a multiple of 11. The system $\begin{cases}11(x+y)=132\\ y-x=3\end{cases}$ has solution $\begin{cases}x=9/2\\ y=15/2\end{cases}$ and $\begin{cases}11(x+y)=143\\ y-x=3\end{cases}$ has solution $\begin{cases}x=5\\ y=8\end{cases}$. The correct answer is $58+85=143$.

Correct Answer: E

11. In the figure, O is the center of the circle. $AB=40$ and $CD=48$ are two chords in the circle. $AB//CD$ and the distance between them is 22. Find the radius of the circle.

A. 26 B. 25 C. 27 D. 24 E. $4\sqrt{39}$

Let x be the length of the radius. $OE\perp AB$ and $OF\perp CD$. Let $OE=y$. Then, $OF=22-y$.

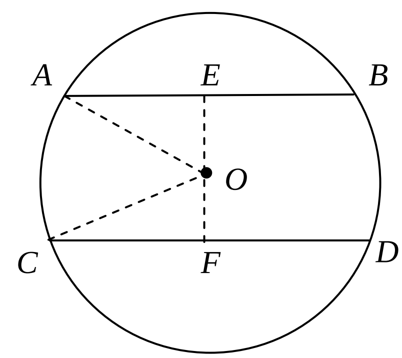
By the Pythagorean Theorem

$$\begin{cases}20^2+y^2=x^2\\ 24^2+(22-y)^2=x^2\end{cases}$$

$$\begin{cases}400+y^2=x^2 & (1)\\ 1060-44y+y^2=x^2 & (2)\end{cases}$$

$(2)-(1)$

$660-44y=0$, $y=15$ and $x=25$.



Correct Answer: B

12. $\sin^{-1}\left(\cos\frac{9\pi}{7}\right)=?$

A. $\frac{2\pi}{7}$ B. $\frac{3\pi}{14}$ C. $-\frac{3\pi}{14}$ D. $-\frac{2\pi}{7}$ E. None of the above

$$\sin^{-1}\left(\cos\frac{9\pi}{7}\right)=\sin^{-1}\left(\cos\left(\frac{3\pi}{2}-\frac{3\pi}{14}\right)\right)=\sin^{-1}\left(\cos\frac{3\pi}{2}\cos\frac{3\pi}{14}+\sin\frac{3\pi}{2}\sin\frac{3\pi}{14}\right)$$

$$=\sin^{-1}\left(-\sin\frac{3\pi}{14}\right)=-\sin^{-1}\left(\sin\frac{3\pi}{14}\right)=-\frac{3\pi}{14}$$

Correct Answer: C

13. Solve the equation $\log_5 x + \log_3 x = 1$.

A. $\left\{\dfrac{\ln 5 \cdot \ln 3}{\ln 15}\right\}$ B. $\left\{e^5\right\}$ C. $\left\{e^{\log_{15}(\ln 5 \cdot \ln 3)}\right\}$ D. $\left\{e^{\log_{15} 5^{\ln 3}}\right\}$ E. None of the above

$$\frac{\ln x}{\ln 5} + \frac{\ln x}{\ln 3} = 1,\ \ln x\left(\frac{1}{\ln 5} + \frac{1}{\ln 3}\right) = 1,\ \ln x\left(\frac{\ln 5 + \ln 3}{\ln 5 \cdot \ln 3}\right) = 1,\ \ln x\left(\frac{\ln 15}{\ln 5 \cdot \ln 3}\right) = 1,$$

$$\ln x = \frac{\ln 5 \cdot \ln 3}{\ln 15} = \frac{\ln 5^{\ln 3}}{\ln 15} = \log_{15} 5^{\ln 3} \text{ and } x = e^{\log_{15} 5^{\ln 3}}.$$

Correct Answer: D

14. The distance between two circles with a common center is 6. If the circumference of the larger circle is 220, find the area of the smaller circle.

A. $\dfrac{12100}{\pi} + 1320 + 36\pi$ B. 208π C. $\dfrac{12100}{\pi} - 1320 + 36\pi$

D. $\dfrac{12100}{\pi} + 36\pi$ E. $\dfrac{12100}{\pi} - 660 + 36\pi$

The radius of the larger circle is $\dfrac{220}{2\pi} = \dfrac{110}{\pi}$ and the radius of the smaller one is $\dfrac{110}{\pi} - 6$. The area of the smaller one is $\pi\left(\dfrac{110}{\pi} - 6\right)^2 = \dfrac{12100}{\pi} - 1320 + 36\pi$.

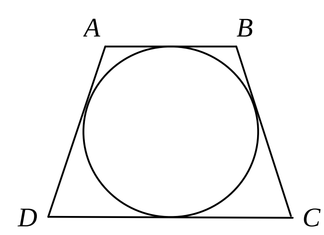
Correct Answer: C

15. A circle is inscribed in a trapezoid *ABCD*. $AB // CD$, *E* and *F* are mid-points of *AD* and *BC* respectively. If the perimeter of the trapezoid is 24, find the length of *EF*.

A. 6 B. 7 C. 8 D. 4 E. Not enough information

Since four sides of the trapezoid are tangent to a circle, we have $AB + CD = AD + BC$.

$$EF = \frac{1}{2}(AB + CD) = \frac{1}{2} \cdot \frac{1}{2}(AB + CD + AD + BC) = \frac{1}{4} \cdot 24 = 6$$



Correct Answer: A

16. Convert the polar equation $r = \dfrac{6\sec\theta}{2\sec\theta - 1}$ to a rectangular equation.

A. $x^2 + y^2 - 12x - 36 = 0$ B. $4x^2 + 3y^2 - 12x - 36 = 0$ C. $3x^2 + 4y^2 + 12x - 36 = 0$
D. $x^2 + y^2 + 12x + 36 = 0$ E. $3x^2 + 4y^2 - 12x - 36 = 0$

$r = \dfrac{6}{2 - \cos\theta}$, $r(2 - \cos\theta) = 6$, $2r - r\cos\theta = 6$, $2r = x + 6$, $4r^2 = x^2 + 12x + 36$,
$4x^2 + 4y^2 = x^2 + 12x + 36$, and $3x^2 + 4y^2 - 12x - 36 = 0$.

Correct Answer: E

17. (Tie Break No. 2) Two sides of a triangle have lengths 2 units and 4 units, and their included angle measures 165°. Find the area of the triangle.

A. $2\sqrt{2 - \sqrt{3}}$ B. $2\left(\sqrt{6} - \sqrt{2}\right)$ C. $\sqrt{6} + \sqrt{2}$ D. $2\sqrt{2 + \sqrt{3}}$ E. None of the above

$$\sin 165^\circ = \sin\frac{330^\circ}{2} = \sqrt{\frac{1 - \cos 330^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

The area of the triangle is $\frac{1}{2} \cdot 2 \cdot 4 \cdot \sin 165^\circ = 2\sqrt{2 - \sqrt{3}}$.

Correct Answer: A

18. Find the value of the sine of an angle between the lines $3x - y + 2 = 0$ and $x + 2y + 1 = 0$.

A. $\dfrac{7}{10}$ B. $\dfrac{7}{5\sqrt{2}}$ C. $-\dfrac{7}{5\sqrt{2}}$ D. $\dfrac{2}{3}$ E. None of the above

Let θ be any angle between the lines. We have $0 < \theta < \pi$ and $\sin\theta = \sin(\pi - \theta) > 0$.
$3x - y + 2 = 0$, $y = 3x + 2$. Vector $\mathbf{v}_1 = \langle 3, 1 \rangle$ is parallel to the line $3x - y + 2 = 0$.
$x + 2y + 1 = 0$, $y = -\frac{1}{2}x - \frac{1}{2}$. Vector $\mathbf{v}_2 = \langle -1, 2 \rangle$ is parallel to the line $x + 2y + 1 = 0$.

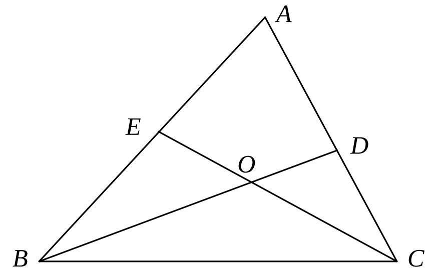
$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{3(-1) + 1(2)}{\sqrt{3^2 + 1^2}\sqrt{(-1)^2 + 2^2}} = \frac{-1}{5\sqrt{2}}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{-1}{5\sqrt{2}}\right)^2} = \sqrt{\frac{49}{50}} = \frac{7}{5\sqrt{2}}$$

Correct Answer: B

19. In ΔABC, $AB = c$, $BC = a$, and $CA = b$. BD and CE are bisectors of $\angle B$ and $\angle C$, respectively. BD and CE intersect at O. Find the ratio $BO : OD$.

A. $(a+c):b$ B. $(a+b):c$ C. $a:(b+c)$ D. $(b+c):a$ E. $b:(a+c)$



In the ΔABC, BD is the bisector of $\angle B$. By the Law of Sines, $\frac{AD}{AB} = \frac{\sin\angle ABD}{\sin\angle ADB} = \frac{\sin\angle DBC}{\sin(\pi - \angle ADB)} = \frac{DC}{BC}$,

$\frac{AD}{DC} = \frac{AB}{BC}$, $\frac{AD+DC}{DC} = \frac{AB+BC}{BC}$,

$\frac{b}{DC} = \frac{c+a}{a}$, and $DC = \frac{ab}{a+c}$.

In the ΔBDC, CO is the bisector of $\angle C$. Similarly, we have $\frac{BO}{OD} = \frac{BC}{DC} = \frac{a}{\frac{ab}{a+c}} = \frac{a+c}{b}$

Correct Answer: A

20. $\cos A \cos 2A \cos 4A \cdots \cos(2^{n-1}A)$ is identically equal to which of the following expressions?

A. $\frac{\sin(2^n A)}{2^{n-1}\cos A}$ B. $\frac{\sin(2^n A)}{2^n \sin A}$ C. $\frac{\cos(2^n A)}{2^{n-1}\cos A}$ D. $\frac{2^n \sin A}{\sin(2^n A)}$ E. None of the above

$$\cos A \cos 2A \cos 4A \cdots \cos(2^{n-1}A) = \frac{1}{2\sin A}\cdot 2\sin A \cos A \cos 2A \cos 4A \cdots \cos(2^{n-1}A)$$

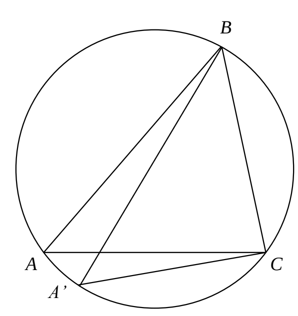
$$= \frac{1}{2^2 \sin A}\cdot 2\sin 2A \cos 2A \cos 4A \cdots \cos(2^{n-1}A) = \frac{1}{2^3 \sin A}\cdot 2\sin 4A \cos 4A \cdots \cos(2^{n-1}A)$$

$$= \cdots = \frac{1}{2^n \sin A}\cdot 2\sin\left(2^{n-1}A\right)\cos\left(2^{n-1}A\right) = \frac{\sin(2^n A)}{2^n \sin A}$$

Correct Answer: B

21. (Tie Break No. 3) ΔABC is inscribed a circle. If $a = 10$ and $\angle A = 15^\circ$, find the radius of the circle.

A. $10\left(\sqrt{6}-\sqrt{2}\right)$ B. $5\left(\sqrt{6}+\sqrt{2}\right)$ C. $5\left(\sqrt{6}-\sqrt{2}\right)$ D. $10\left(\sqrt{6}+\sqrt{2}\right)$ E. $10\left(\sqrt{3}-1\right)$



Let BA' be a diameter of the circle.

$\angle BA'C = \angle A$ and $\angle A'CB = 90^\circ$

$$BA' = \frac{BC}{\sin\angle BA'C} = \frac{a}{\sin\angle A} = \frac{10}{\sin 15^\circ} = \frac{10}{\frac{\sqrt{6}-\sqrt{2}}{4}} = 10\left(\sqrt{6}+\sqrt{2}\right)$$

Therefore, the radius is $5\left(\sqrt{6}+\sqrt{2}\right)$.

Correct Answer: B

22. The sum of the solutions of equation $z-2|z|=-7+4i$ is

A. $\frac{5}{3}+4i$ B. $3+\frac{8}{3}i$ C. $\frac{14}{3}+4i$ D. $\frac{14}{3}+8i$ E. $8+\frac{14}{3}i$

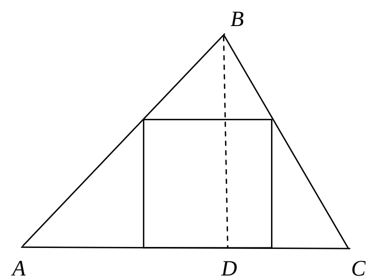
Let $z=x+yi$. Then $|z|=\sqrt{x^2+y^2}$ Then we have the system $\begin{cases} x-2\sqrt{x^2+y^2}=-7 \\ y=4 \end{cases}$.

By substitution, $x-2\sqrt{x^2+16}=-7$. Solve the equation, $x=3$ or $x=\frac{5}{3}$. The solution set of $z-2|z|=-7+4i$ is $\left\{3+4i,\ \frac{5}{3}+4i\right\}$ and the sum of the solutions is $\frac{14}{3}+8i$.

Correct Answer: D

23. In the figure, the three sides of ΔABC have the lengths $a=13$, $b=14$, and $c=15$. Find the perimeter of the square inscribed in the triangle.

A. $336/13$ B. $84/13$ C. 36 D. $172/7$ E. $364/15$



Let $BD\perp AC$ and $BD=h$. By Heron's formula, area of the triangle is $S=\sqrt{21(21-13)(21-14)(21-15)}=84$. We also have $S=\frac{1}{2}bh$. Then $h=\frac{2S}{b}=\frac{2(84)}{14}=12$.

Let the length of the sides of the square be x. The area of the triangle is also equal to the sum of areas of the square and three small triangles.

$$x^2+\frac{1}{2}(b-x)x+\frac{1}{2}x(h-x)=x^2+\frac{1}{2}(14-x)x+\frac{1}{2}x(12-x)=13x$$

$13x=84$ and the perimeter of the square is $4x=\frac{336}{13}$.

Correct Answer: A

24. A fifth degree polynomial $p(x)$ with real coefficients has complex zeros $x=1\pm i\sqrt{2}$, $x=\pm 3$, and $x=\frac{2}{3}$. If the y-intercept of its graph is $(0,9)$, find the coefficient of its x^4 term.

A. $4/3$ B. $-4/3$ C. -4 D. $-8/3$ E. The answer is not unique.

$$p(x)=k\left(x-1+i\sqrt{2}\right)\left(x-1-i\sqrt{2}\right)(x+3)(x-3)\left(x-\frac{2}{3}\right)$$

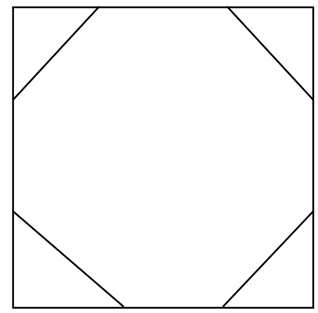
$p(0)=9$ implies $p(0)=k\left(-1+i\sqrt{2}\right)\left(-1-i\sqrt{2}\right)(3)(-3)\left(-\frac{2}{3}\right)=9$ and $k=\frac{1}{2}$.

The coefficient of x^4 term of $p(x)$ is $c_4=-\frac{1}{2}\left(\left(1+i\sqrt{2}\right)+\left(1-i\sqrt{2}\right)+3+(-3)+\frac{2}{3}\right)=-\frac{4}{3}$.

Correct Answer: B

25. A regular octagon is obtained by cutting four equal right isosceles triangles from four corners of a square. If the length of the sides of the square is 1, find the area of the octagon.

A. $2\left(\sqrt{2}-1\right)$ B. $\dfrac{1}{1+\sqrt{2}}$ C. $\dfrac{1+\sqrt{2}}{2+\sqrt{2}}$ D. $1-\dfrac{\sqrt{2}}{4}$ E. $\sqrt{3}-1$



Let x be the length of the sides of the octagon.

$x=\left(1-2\left(x\dfrac{\sqrt{2}}{2}\right)\right)$, $\left(\sqrt{2}+1\right)x=1$, and $x=\dfrac{1}{\sqrt{2}+1}=\sqrt{2}-1$.

The area of one right triangle is

$S_T=\dfrac{1}{2}\left(\dfrac{\sqrt{2}}{2}x\right)^2=\dfrac{1}{4}\left(\sqrt{2}-1\right)^2=\dfrac{3-2\sqrt{2}}{4}$.

The area of the octagon is $S=1-4S_T=1-\left(3-2\sqrt{2}\right)=2\left(\sqrt{2}-1\right)$.

Correct Answer: A

26. If $\tan\left(\dfrac{\pi}{4}+A\right)=2$, find the value of $2\cos^2 A-\sin 2A$.

A. 5/6 B. 6/5 C. $\sqrt{3}/4$ D. $4/\sqrt{3}$ E. 8/7

$\tan\left(\dfrac{\pi}{4}+A\right)=\dfrac{1+\tan A}{1-\tan A}=2$ implies $\tan A=\dfrac{1}{3}$.

$$2\cos^2 A-\sin 2A=\frac{2\cos^2 A-2\sin A\cos A}{\cos^2 A+\sin^2 A}=\frac{2(1-\tan A)}{1+\tan^2 A}=\frac{2\left(1-\frac{1}{3}\right)}{1+\left(\frac{1}{3}\right)^2}=\frac{6}{5}$$

Correct Answer: B

27. Which of the following trigonometric expressions is identical to $\sin^2 A-\sin^2 B$?

A. $\cos^2 A-\cos^2 B$ B. $\sin(A+B)\sin(A-B)$ C. $\cos(A+B)\cos(A-B)$
D. $\left(\sin A-\sin B\right)^2$ E. $\left(\cos A-\cos B\right)^2$

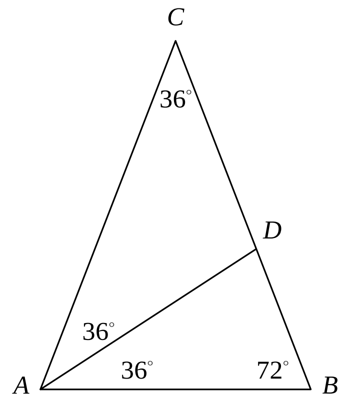
$$\sin^2 A-\sin^2 B=\sin^2 A\left(\cos^2 B+\sin^2 B\right)-\sin^2 B\left(\cos^2 A+\sin^2 A\right)$$
$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B=\left(\sin A\cos B+\cos A\sin B\right)\left(\sin A\cos B-\cos A\sin B\right)$$
$$=\sin\left(A+B\right)\sin\left(A-B\right)$$

Correct Answer: B

28. Find the exact value of $\sin 72^\circ$

A. $\frac{\sqrt{5}-1}{4}$ B. $\frac{\sqrt{10-2\sqrt{5}}}{4}$ C. $\frac{\sqrt{10+2\sqrt{5}}}{4}$ D. $\frac{\sqrt{5}+1}{4}$ E. $\frac{\sqrt{6}-\sqrt{2}}{4}$

Consider the isosceles ΔABC with $\angle C = 36^\circ$, $\angle B = \angle A = 72^\circ$, and $AB = 1$. Let AD be the bisector of $\angle A$. Then $AB = AD = CD = 1$.



Since $\Delta CAB \sim \Delta ABD$, $\frac{CB}{AB} = \frac{AB}{DB} = \frac{AB}{CB-CD}$.

$\frac{CB}{1} = \frac{1}{CB-1}$ and $(CB)^2 - CB - 1 = 0$. $CB = \frac{1+\sqrt{5}}{2}$

$$\cos 72^\circ = \cos B = \frac{\frac{1}{2}AB}{CB} = \frac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4}$$

$$\sin 72^\circ = \sqrt{1-\cos^2 72^\circ} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

Correct Answer: C

29. If $\{a_n\}(a_n > 0)$ is a geometric sequence and $a_5 a_6 = 9$, find the value of the sum $\log_3 a_1 + \log_3 a_2 + \cdots + \log_3 a_{10}$

A. 12 B. 8 C. 10 D. $2 + 10\log_3 2$ E. $2 + \log_3 5$

$$a_1 a_{10} = a_2 a_9 = a_3 a_8 = a_4 a_7 = a_5 a_6 = 9$$
$$\log_3 a_1 + \log_3 a_2 + \cdots + \log_3 a_{10} = \log_3 a_1 a_{10} + \log_3 a_2 a_9 + \cdots + \log_3 a_5 a_6 = 5\log_3 9 = 10$$

Correct Answer: C

30. Solve the inequality $2 + \log_{\frac{1}{2}}(5-x) + \log_2 \frac{1}{x} > 0$

A. $(4,5)$ B. $(-\infty,1)$ C. $(-\infty,1)\cup(4,\infty)$ D. $(4,\infty)$ E. $(0,1)\cup(4,5)$

We have to have $x > 0$ and $5 - x > 0$. Hence, $0 < x < 5$. For such x-values, the inequality is equivalent to the inequality $\log_{\frac{1}{2}}\frac{1}{4} + \log_{\frac{1}{2}}(5-x) - \log_{\frac{1}{2}}\frac{1}{x} = \log_{\frac{1}{2}}\frac{1}{4}x(5-x) > 0$

Then, $\frac{1}{4}x(5-x) < 1$, $x^2 - 5x + 4 > 0$, $(x-4)(x-1) > 0$, $x < 1$ or $4 < x$.

The solution set is $(0,1)\cup(4,5)$.

Correct Answer: E

31. Find the domain of the inverse function of $f(x)=\dfrac{e^x-1}{e^x+1}$.

A. $(-\infty,\ \infty)$ B. $(-1,1)$ C. $(-\infty,-1)\cup(1,\ \infty)$ D. $[-1,1]$ E. $(0,1)$

Solve $x=\dfrac{e^y-1}{e^y+1}$ for y.

$xe^y+x=e^y-1$, $e^y-xe^y=1+x$, $e^y=\dfrac{1+x}{1-x}$, and $y=\ln\dfrac{1+x}{1-x}$.

Solve the inequality $\dfrac{1+x}{1-x}>0$, $-1<x<1$.

Correct Answer: B

32. In the figure, AB is a diameter of a semicircle. D and E are two points on the semicircle. AD and BE intersect at the point C. If $AB=16$, $DC=3$ and $EC=2$, find the perimeter of the ΔABC.

A. 38 B. 36 C. $16(\sqrt{2}+1)$ D. $12\sqrt{3}+16$ E. 40

$\angle D=\angle E=90^\circ$. Let $AC=x$ and $BC=y$.

Since $\Delta BDC\sim\Delta AEC$, $\dfrac{y}{x}=\dfrac{BC}{AC}=\dfrac{DC}{EC}=\dfrac{3}{2}$ and $y=\dfrac{3}{2}x$.

By the Pythagorean's Theorem,

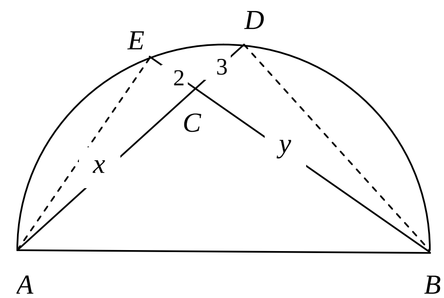
$(AB)^2-(BE)^2=(AE)^2=(AC)^2-(EC)^2$,

$16^2-(y+2)^2=x^2-2^2$, and $16^2-\left(\dfrac{3}{2}x+2\right)^2=x^2-2^2$.

We have $13x^2+24x-1024=0$, $(13x+128)(x-8)=0$,

$x=8$, and $y=12$. The perimeter of the ΔABC is

$8+12+16=36$.



Correct Answer: B

33. Write the expression $i^{2011}+i^{2010}+\cdots+i+1$ in standard form $a+bi$

A. 0 B. $1+i$ C. 1 D. i E. $1-i$

$$i^{2011}+i^{2010}+\cdots+i+1=\frac{1-i^{2012}}{1-i}=\frac{1-1}{1-i}=\frac{0}{1-i}=0$$

Correct Answer: A

34. Solve the inequality $\sqrt{2x+3} > x+1$

A. $\left[-\frac{3}{2},-1\right]$ B. $[-1,\sqrt{2})$ C. $(-\sqrt{2},\sqrt{2})$ D. $\left[-\frac{3}{2},\sqrt{2}\right)$ E. $\left[-\frac{3}{2},-1\right)$

The solution set of the inequality is the union of the solution sets of the systems

$$\begin{cases} x+1<0 \\ 2x+3\geq 0 \end{cases} \text{ and } \begin{cases} x+1\geq 0 \\ 2x+3\geq 0 \\ 2x+3>x^2+2x+1 \end{cases}.$$

The solution set of the first system is $\left[-\frac{3}{2},-1\right)$ and the solution set of the second system $[-1,\sqrt{2})$. Their union is $\left[-\frac{3}{2},\sqrt{2}\right)$

Correct Answer: D

35. If the domain of function $f(x)=(x-1)^{\frac{2}{3}}(ax^2+4ax+3)^{-1}$ is the set of all real numbers, then what are the permissible values of the parameter a?

A. $(0, 3/4)$ B. $(-\infty, \infty)$ C. $(3/4, \infty)$ D. $[0, 3/4)$ E. $(-\infty, 0)$

When $a=0$, the domain of the function is $(-\infty, \infty)$. When $a\neq 0$, $ax^2+4ax+3$ does not have real zeros if $(4a)^2-4a(3)<0$. $4a(4a-3)<0$ and $0<a<\frac{3}{4}$. The range of real parameter a is $0\leq a<\frac{3}{4}$.

Correct Answer: D

36. $\{a_n\}_{n=1}^{\infty}$ is an arithmetic sequence. $a_7=17$ and $a_{31}=65$. Find $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}}$.

A. ∞ B. 2 C. 2/5 D. 1/10 E. 1

Let d be the common difference of the sequence.

Solve the system $\begin{cases} a_7=a_1+6d=17 \\ a_{31}=a_1+30d=65 \end{cases}$. We have $a_1=5$ and $d=2$.

$$\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}}=\sum_{k=1}^{n}\frac{1}{(a_1+(k-1)d)(a_1+kd)}=\sum_{k=1}^{n}\frac{1}{d}\left(\frac{1}{a_1+(k-1)d}-\frac{1}{a_1+kd}\right)=\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_1+nd}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}}=\lim_{n\to\infty}\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_1+nd}\right)=\lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{5}-\frac{1}{5+2n}\right)=\frac{1}{10}$$

Correct Answer: D

37. Which of the following is the equation of the circle such that its center is on the parabola $y^2 = 2x$ and it is tangent to both the *x*-axis and directrix of the parabola?

A. $x^2 + y^2 - x - 2y - \frac{1}{4} = 0$ B. $x^2 + y^2 + x - 2y + 1 = 0$ C. $x^2 + y^2 - x - 2y + \frac{1}{4} = 0$

D. $x^2 + y^2 - x - 2y + 1 = 0$ E. $x^2 + y^2 - x + 2y - \frac{1}{4} = 0$

The directrix of the parabola is $x = -\frac{1}{2}$. The circle is tangent to both the *x*-axis $y = 0$ and the directrix. Its center also is on the line $y = x + \frac{1}{2}$ or $y = -x - \frac{1}{2}$.

Solve the systems $\begin{cases} y^2 = 2x \\ y = x + \frac{1}{2} \end{cases}$ and $\begin{cases} y^2 = 2x \\ y = -x - \frac{1}{2} \end{cases}$. The center of the circle is $\left(\frac{1}{2}, 1\right)$ or $\left(\frac{1}{2}, -1\right)$ and its radius $r = 1$. The equation of the circle is $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = 1$ or $\left(x - \frac{1}{2}\right)^2 + (y+1)^2 = 1$. That is $x^2 + y^2 - x - 2y + \frac{1}{4} = 0$ or $x^2 + y^2 - x + 2y + \frac{1}{4} = 0$.

Correct Answer: C

38. (Tie Break No. 4) Find the sum of all positive rational numbers that are less than 10 and that have denominator 10 when written in lowest terms.

A. 250 B. 200 C. 180 D. 210 E. 195

A positive rational number that is less than 10 and has denominator 10 when written in lowest terms can be written in the form $\frac{10n + r}{10}$, where *n* and *r* are integers such that $n \in S_n = \{0, 1, 2, \cdots, 9\}$ and $r \in S_r = \{1, 3, 7, 9\}$. The sum of all such fractions is

$$\left(\frac{1}{10} + \frac{3}{10} + \frac{7}{10} + \frac{9}{10}\right) + \left(\frac{10+1}{10} + \frac{10+3}{10} + \frac{10+7}{10} + \frac{10+9}{10}\right) + \cdots + \left(\frac{90+1}{10} + \frac{90+3}{10} + \frac{90+7}{10} + \frac{90+9}{10}\right)$$

$$= \frac{20}{10} + \left(4 \cdot 1 + \frac{20}{10}\right) + \left(4 \cdot 2 + \frac{20}{10}\right) + \cdots + \left(4 \cdot 9 + \frac{20}{10}\right) = 4 \cdot \frac{(1+9)9}{2} + 2 \cdot 10 = 200 .$$

Correct Answer: B

39. If the line tangent to the graph of the function $f(x)$ at the point $(1,7)$ passes through the point $(-2,-2)$, then $f'(1)$ is

A. -5 B. 1 C. 3 D. 7 E. Undefined

The slope of the tangent line to the graph is $m=\dfrac{7-(-2)}{1-(-2)}=3=f'(1)$.

Correct Answer: C

40. Urn #1 contains 10 balls: 4 red and 6 blue. Urn #2 contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Determine the number of blue balls in urn #2.

A. 4 B. 20 C. 24 D. 44 E. 64

Let x be the number of blue balls in the urn #2. The probability that both balls are the same color is $\dfrac{4}{10}\cdot\dfrac{16}{16+x}+\dfrac{6}{10}\cdot\dfrac{x}{16+x}=0.44$.

$32+3x=0.44(5(16+x))$, $32+3x=35.2+2.2x$, $0.8x=3.2$, and $x=4$.

Correct Answer: A

41. Find the real solution of the equation $4^x+6^x=9^x$

A. $\ln\dfrac{-1+\sqrt{5}}{2}$ B. $\dfrac{\ln(\sqrt{5}-1)-\ln 2}{\ln 2-\ln 3}$ C. $\log_{\frac{2}{3}}\dfrac{-1\pm\sqrt{5}}{2}$ D. $\ln\dfrac{\sqrt{5}-1}{2}-\ln\dfrac{2}{3}$ E. $\dfrac{\ln(\sqrt{5}-1)}{\ln 3}$

The equation can be rewritten in $\left(\dfrac{4}{9}\right)^x+\left(\dfrac{6}{9}\right)^x=1$ or $\left(\dfrac{2}{3}\right)^{2x}+\left(\dfrac{2}{3}\right)^x=1$. $\left(\dfrac{2}{3}\right)^x=\dfrac{-1\pm\sqrt{5}}{2}$.

Since $\left(\dfrac{2}{3}\right)^x>0$, $\left(\dfrac{2}{3}\right)^x=\dfrac{-1+\sqrt{5}}{2}$, $x\ln\dfrac{2}{3}=\ln\dfrac{\sqrt{5}-1}{2}$. $x=\dfrac{\ln\dfrac{\sqrt{5}-1}{2}}{\ln\dfrac{2}{3}}=\dfrac{\ln(\sqrt{5}-1)-\ln 2}{\ln 2-\ln 3}$.

Correct Answer: B

42. Find the coefficient of the x^3 term in the expansion of $(1+x)^{10}(1-x)^8$

A. -8 B. -4 C. -16 D. 15 E. -12

$(1+x)^{10}(1-x)^8=(1+x)^2(1+x)^8(1-x)^8=(1+2x+x^2)(1-x^2)^8$

$=(1+2x+x^2)(1-8x^2+28x^4-\cdots-8x^{14}+x^{16})$

In the expansion, the coefficient of x^3 term is $2\cdot(-8)=-16$.

Correct Answer: C

43. $\int_a^c f(x)dx = \ln 32$ and $\int_a^c |f(x)|dx = \ln 64$. $f(x) \geq 0$ in (a, b) and $f(x) < 0$ in (b, c). Find the value of $\int_b^c f(x)dx$

A. $-\ln\sqrt{2}$ B. $\ln(1/2)$ C. $-\ln 8$ D. $-\ln 32$ E. $-\ln(1/2)$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx = \ln 32 \text{ and } \int_a^c |f(x)|dx = \int_a^b f(x)dx - \int_b^c f(x)dx = \ln 64$$

$$\int_b^c f(x)dx = \frac{1}{2}(\ln 32 - \ln 64) = \frac{1}{2}\ln\frac{32}{64} = -\frac{1}{2}\ln 2 = -\ln\sqrt{2}$$

Correct Answer: A

44. Which of the following series diverge?

I. $\sum_{n=0}^{\infty}\left(\frac{\sin 2}{\pi}\right)^n$ II. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n}}\right)$ III. $\sum_{n=1}^{\infty}\left(\frac{e^n}{e^n+1}\right)$

A. III only B. I and II only C. I and III only D. II and III only E. I, II, and III

The series I is a geometric series with common ratio $r = \frac{\sin 2}{\pi}$. Since $\left|\frac{\sin 2}{\pi}\right| < \frac{1}{3}$, it is convergent. For the series II, we have $\frac{1}{n} < \frac{1}{\sqrt[3]{n}}$. It is divergent by the comparison test.

For the series III, $\lim_{n\to\infty}\frac{e^n}{e^n+1} = \lim_{n\to\infty}\frac{1}{1+e^{-n}} = 1 > 0$. It is divergent by the divergent test.

Correct Answer: D

45. (Tie Break No. 5) $\cot^{-1}\left(\frac{43}{32}\right) - \tan^{-1}\left(\frac{1}{4}\right) = ?$

A. $\tan^{-1}(5/4)$ B. $\cos^{-1}(3/5)$ C. $\sin^{-1}(-3/4)$ D. $\cos^{-1}(12/13)$ E. $\tan^{-1}(12/5)$

Let $A = \cot^{-1}\left(\frac{43}{32}\right)$ and $B = \tan^{-1}\left(\frac{1}{4}\right)$. $\tan A = \frac{32}{43} > \frac{1}{4} = \tan B$ implies $A > B$. $0 < A - B < \frac{\pi}{2}$. $A - B \neq \sin^{-1}(-3/4) < 0$. The choice C can not be the answer.

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{32}{43} - \frac{1}{4}}{1 + \frac{32}{43}\cdot\frac{1}{4}} = \frac{5}{12},\ \cos(A-B) = \frac{12}{\sqrt{12^2 + 5^2}} = \frac{12}{13}.$$

Correct Answer: D