1. For any positive integer n, $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$. Determine the value of n if $n! = 30(n-2)!$

A. 7 B. 5 C. 10 D. 6 E. 9

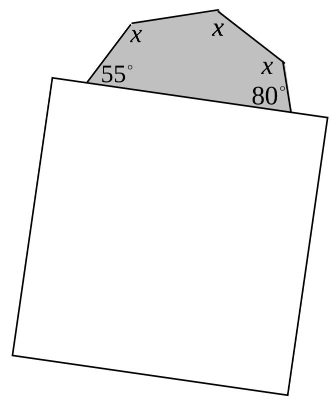
$n! = 30(n-2)!$, $n(n-1)\cdot(n-2)! = 30(n-2)!$, $n(n-1) = 30$, $n^2 - n - 30 = 0$, $(n+5)(n-6) = 0$ and $n = 6$.

Correct answer: D

2. The picture below shows a regular polygon in gray that is partially covered by a white sheet of paper. How many sides does the polygon have?

A. 6 B. 8 C. 10 D. 12
E. The answer cannot be determined from the given information

Let x be the degree measure of an interior angle of the regular polygon.
In the figure, $3x + 55^\circ + 80^\circ = (5-2)\cdot 180^\circ$,
$3x = 540^\circ - 135^\circ = 405^\circ$, and $x = 135^\circ$.
Let n be number of sides of the regular polygon.
$135^\circ n = 180^\circ(n-2)$, $360^\circ = 45^\circ n$, and $n = 8$.



Correct answer: B

3. If the graph of $2x + y + 3 = 0$ is perpendicular to the graph of $3x + ky + 2 = 0$, then k equals what?

A. -6 B. $2/3$ C. 6 D. $-2/3$ E. $-3/2$

Solve $2x + y + 3 = 0$ for y. $y = -2x - 3$. Its slope is $m_1 = -2$. Solve $3x + ky + 2 = 0$ for y.

$y = -\frac{3}{k}x - \frac{2}{k}$

Its slope is $m_2 = -\frac{3}{k}$.

$m_1 m_2 = (-2)\left(-\frac{3}{k}\right) = -1$ and $k = -6$.

Correct answer: A

4. $\dfrac{4-8x}{x^4+4x^2}$ is identical to which of the following expressions?

A. $\dfrac{2x}{x^2+4}-\dfrac{3}{x^2+4}-\dfrac{2}{x}+\dfrac{1}{x^2}$ B. $\dfrac{x}{x^2+4}-\dfrac{1}{x^2+4}-\dfrac{3}{x}+\dfrac{1}{x^2}$ C. $\dfrac{2x}{x^2+4}-\dfrac{1}{x^2+4}-\dfrac{2}{x}+\dfrac{1}{x^2}$

D. $\dfrac{2x}{x^2+4}-\dfrac{1}{x^2+4}-\dfrac{2}{x}-\dfrac{3}{x^2}$ E. $\dfrac{x}{x^2+4}-\dfrac{1}{x^2+4}-\dfrac{2}{x}-\dfrac{1}{x^2}$

Let $\dfrac{4-8x}{x^4+4x^2}=\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x}+\dfrac{D}{x^2}$. We have $4-8x=(Ax+B)x^2+Cx(x^2+4)+D(x^2+4)$

Letting $x=0$, we have $4=4D$ and $D=1$.

Letting $x=2i$, we have $4-8(2i)=(2Ai+B)(2i)^2$, $4-16i=-4B-8Ai$, $B=-1$, and $A=2$.

Then we have $4-8x=(2x-1)x^2+Cx(x^2+4)+(x^2+4)$.

Letting $x=1$ in the above, we have $-4=1+5C+5$ and $C=-2$.

Therefore, $\dfrac{4-8x}{x^4+4x^2}=\dfrac{2x}{x^2+4}-\dfrac{1}{x^2+4}-\dfrac{2}{x}+\dfrac{1}{x^2}$.

Correct answer: C

5. In the diagram below, the radius of each circle is ½ the radius of the next largest circle. If a point inside the largest circle is chosen at random, what is the probability that it would lie in the shaded region?

A. 1/2 B. 3/16 C. 7/16 D. 13/64 E. 27/64

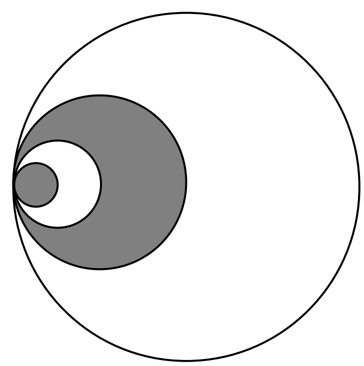
Let r be the radius of the largest circle. Its area is πr^2.

The area of the shaded region is

$$\pi\left(\frac{r}{2}\right)^2-\pi\left(\frac{r}{4}\right)^2+\pi\left(\frac{r}{8}\right)^2=\pi\left(\frac{1}{4}-\frac{1}{16}+\frac{1}{64}\right)r^2=\frac{13}{64}\pi r^2.$$

The probability that point would lie in the shaded region is

$$P=\frac{\frac{13}{64}\pi r^2}{\pi r^2}=\frac{13}{64}.$$



Correct answer: D

6. Find the minimum value of the following function of *x* and *y*:
$f(x,y)=x^2-6xy+10y^2-10y+40$

A. 40 B. 15 C. 10 D. 5 E. -5

$$f(x,y)=x^2-6xy+9y^2+y^2-10y+25+15=(x-3y)^2+(y-5)^2+15$$

This shows $f(x,y)$ has a minimum value 15 at $(x,y)=(15,5)$.

Correct answer: B

7. If x and y are the smallest positive possible angles for which $\sin x = \frac{1}{\sqrt{5}}$ and $\sin y = \frac{1}{\sqrt{10}}$, then find the value of $(x+y)$.

A. $\pi/6$ B. $\pi/5$ C. $\pi/4$ D. $\pi/3$ E. $\pi/2$

$0 < x, y < \pi/2$. $\sin x = \frac{1}{\sqrt{5}}$ implies $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2} = \frac{2}{\sqrt{5}}$ and $\sin y = \frac{1}{\sqrt{10}}$ implies $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2} = \frac{3}{\sqrt{10}}$.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y = \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$x + y = \pi/4$

Correct answer: C

8. Evaluate $\frac{1}{\log_4 6} + \frac{1}{\log_9 6}$

A. 2 B. 4 C. 6 D. 9 E. None of the above

$$\frac{1}{\log_4 6} + \frac{1}{\log_9 6} = \frac{1}{\frac{\log 6}{\log 4}} + \frac{1}{\frac{\log 6}{\log 9}} = \frac{\log 4}{\log 6} + \frac{\log 9}{\log 6} = \frac{\log(4\cdot 9)}{\log 6} = \frac{\log 36}{\log 6} = \frac{\log 6^2}{\log 6} = \frac{2\log 6}{\log 6} = 2$$

Correct answer: A

9. (Tie Break No.1) In trapezoid $ABCD$, $AB \,//\, CD$, $AB = 3$, $CD = 6$, and $AD = BC = 5$. Find the length of the diagonal AC.

A. $3\sqrt{5}$ B. $\sqrt{43}$ C. $5\sqrt{2}$ D. 7 E. None of the above

$DF = EC$ since $\Delta ADF \cong \Delta BCE$.

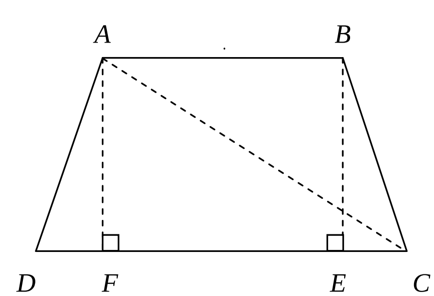
$$\cos\angle D = \frac{(DF)}{(AD)} = \frac{\left(\frac{(CD)-(AB)}{2}\right)}{(AD)} = \frac{\left(\frac{6-3}{2}\right)}{5} = \frac{3}{10}$$

By the Law of Cosines,

$$(AC)^2 = (DA)^2 + (DC)^2 - 2(DA)(DC)\cos\angle D$$

$$= 5^2 + 6^2 - 2(5)(6)\cdot\frac{3}{10} = 43.$$

$AC = \sqrt{43}$



Correct answer: B

10. If $\dfrac{2^{2011}+2^{2008}}{2^{2010}-2^{2009}}$ is written in the form $\dfrac{a}{b}$ where a and b are integers with no common divisors, what is $a+b$?

A. 11 B. 2 C. 12 D. 14 E. 38

$$\frac{2^{2011}+2^{2008}}{2^{2010}-2^{2009}}=\frac{2^{2008}(2^3+1)}{2^{2009}(2-1)}=\frac{(2^3+1)}{2\cdot 1}=\frac{9}{2}=\frac{a}{b} \text{ and } a+b=9+2=11$$

Correct answer: A

11. The product of two positive integers is 25 times their quotient. What can you say for sure about this situation?

A. The sum of the numbers is at least 10.

B. The difference of the numbers is at most 10.

C. Nothing can be said for sure.

D. One of the numbers is 5.

E. None of the above.

Let x and y be the positive integers. $xy=25\dfrac{x}{y}$, $xy^2=25x$, $y^2=25$, and $y=5$.

Correct answer: D

12. Evaluate: $\left[2\left(\cos\left(\dfrac{5\pi}{18}\right)+i\sin\left(\dfrac{5\pi}{18}\right)\right)\right]^6$

A. $32+32i\sqrt{3}$ B. $32-32i\sqrt{3}$ C. $32\sqrt{3}+32i$ D. $32\sqrt{3}-32i$ E. $16\sqrt{3}+16i$

$$\left[2\left(\cos\left(\frac{5\pi}{18}\right)+i\sin\left(\frac{5\pi}{18}\right)\right)\right]^6=2^6\left[\cos\left(6\cdot\frac{5\pi}{18}\right)+i\sin\left(6\cdot\frac{5\pi}{18}\right)\right]=64\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=64\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$
$$=32-32i\sqrt{3}$$

Correct answer: B

13. What is the distance between the points of intersection of the graphs of $y = x^2 - 4x - 3$ and $x = y + 3$?

A. $\sqrt{5}$ B. $5\sqrt{2}$ C. $\sqrt{10}$ D. $\sqrt{15}$ E. The graphs only intersect at one point.

Substitute $y = x - 3$ into $y = x^2 - 4x - 3$. $x - 3 = x^2 - 4x - 3$, $x^2 - 5x = 0$, $x(x-5) = 0$, $x = 0$ or $x = 5$.
The graphs intersect at $(0, -3)$ and $(5, 2)$.
The distance between them is $\sqrt{(5-0)^2 + (2-(-3))^2} = \sqrt{25+25} = 5\sqrt{2}$.

Correct answer: B

14. If $\cos x = \frac{3}{5}$ and $\cot x < 0$, find the value of $\frac{\sin x - \tan x}{1 + \sec x}$.

A. $\frac{4}{5}$ B. $-\frac{4}{5}$ C. $-\frac{1}{5}$ D. $\frac{1}{5}$ E. None of the above

$\cos x = \frac{3}{5} > 0$ and $\cot x < 0$ imply $\sin x < 0$. $\sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{1 - \left(\frac{3}{5}\right)^2} = -\frac{4}{5}$.

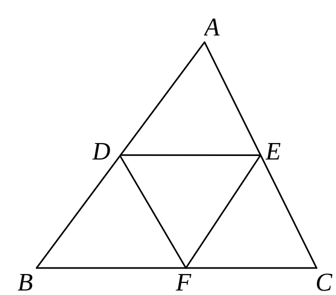
$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$ and $\sec x = \frac{1}{\cos x} = \frac{5}{3}$. Then $\frac{\sin x - \tan x}{1 + \sec x} = \frac{-\frac{4}{5} - \left(-\frac{4}{3}\right)}{1 + \frac{5}{3}} = \frac{\frac{8}{15}}{\frac{8}{3}} = \frac{3}{15} = \frac{1}{5}$

Correct answer: D

15. In ΔABC, D, E, and F are mid-points of AB, BC, and CA, respectively. Find the ratio of the area of ΔDEF to the area of ΔABC.

A. $1:4$ B. $1:2$ C. $1:3$ D. $2:3$ E. None of the above

Let $AB = c$, $BC = a$, and $CA = b$. Then $DE = \frac{1}{2}a, EF = \frac{1}{2}c$, and $FD = \frac{1}{2}b$.



Let s be the half of the perimeter of the ΔABC, By Heron's formula, the area of ΔABC is $\sqrt{s(s-a)(s-b)(s-c)}$. The half of the perimeter of the ΔDEF is $\frac{s}{2}$ and the area of ΔDEF is $\sqrt{\frac{s}{2}\left(\frac{s}{2} - \frac{a}{2}\right)\left(\frac{s}{2} - \frac{b}{2}\right)\left(\frac{s}{2} - \frac{c}{2}\right)} = \frac{1}{4}\sqrt{s(s-a)(s-b)(s-c)}$. Therefore, their ratio is $1:4$.

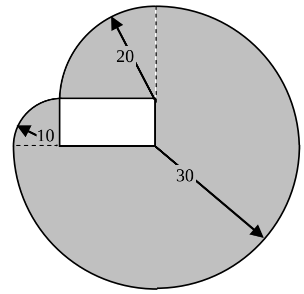
Correct answer: A

16. A cow is tied with a 30 foot rope to an outside corner of a rectangular barn having dimensions 20 feet by 10 feet. How many square feet of grazing area does the cow have?

A. 800π B. 200 C. 900π D. 675π E. 775π

The grazing area the cow has is the shaded region in the figure.
It is the following sum.

$$\frac{3}{4}\cdot\pi\cdot 30^2+\frac{1}{4}\cdot\pi\cdot 10^2+\frac{1}{4}\cdot\pi\cdot 20^2=675\pi+25\pi+100\pi=800\pi$$

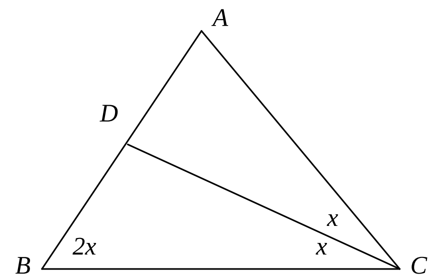


Correct answer: A

17. In ΔABC, $AB=AC$ and CD is the bisector of $\angle C$. If $\angle BDC=75^\circ$, find $\angle A$.

A. 40° B. 50° C. 35° D. 60° E. 55°

Let $\angle DCB=\angle ACD=x$.
$\angle B=\angle ACB=2x$
$\angle B+\angle DCB+\angle BDC=2x+x+75^\circ=180^\circ$ and $x=35^\circ$.
$\angle A=180^\circ-4x=180^\circ-140^\circ=40^\circ$



Correct answer: A

18. (Tie Break No. 2) Given $f(0)=3$; $f(n+1)=2f(n)+3$. What is $f(10)$?

A. 6771 B. 6241 C. 7142 D. 5763 E. 6141

$f(1)=2f(0)+3$,
$f(2)=2f(1)+3=2(2f(0)+3)+3=2^2f(0)+2\cdot3+3$
$f(3)=2f(2)+3=2(2^2f(0)+2\cdot3+3)+3=2^3f(0)+2^2\cdot3+2\cdot3+3$
……
By math induction, we have

$$f(n)=2^nf(0)+2^{n-1}\cdot3+2^{n-2}\cdot3+\cdots+2\cdot3+3=2^nf(0)+\frac{2^n-1}{2-1}\cdot3.$$

$$f(10)=2^{10}\cdot3+(2^{10}-1)\cdot3=6141$$

Correct answer: E

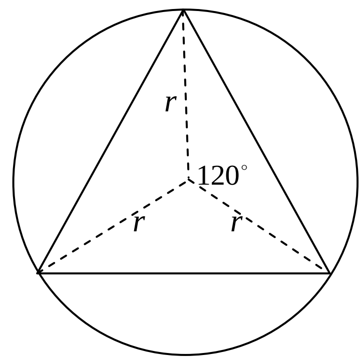
19. Find the area of an equilateral triangle that is inscribed in a circle if the circumference of the circle is 6π.

A. $\frac{8\sqrt{3}}{3}$ B. $\frac{27\sqrt{3}}{4}$ C. $\frac{3\sqrt{2}}{2}$ D. $\frac{30\sqrt{5}}{7}$ E. $\frac{5\sqrt{3}}{4}$

The radius of the circle is $r = \frac{C}{2\pi} = \frac{6\pi}{2\pi} = 3$.

The area of the triangle is

$A = 3\left(\frac{1}{2}r^2 \sin 120^\circ\right) = 3\left(\frac{1}{2}\cdot 3^2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{27\sqrt{3}}{4}$.



Correct answer: B

20. Suppose x is a complex number satisfying the equation $x + \frac{1}{x} = 1$. What is the value of $x^3 + \frac{1}{x^3}$?

A. -2 B. -1 C. 0 D. 1 E. 2

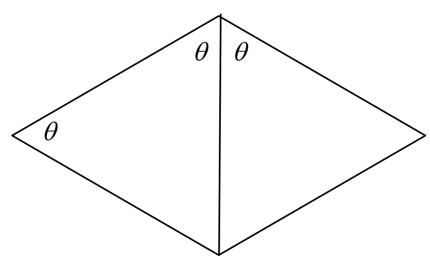
$\left(x+\frac{1}{x}\right)^3 = 1^3$, $x^3 + 3\cdot x^2 \cdot \frac{1}{x} + 3\cdot x \cdot \left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 = 1$, $x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = 1$, $x^3 + 3\cdot 1 + \frac{1}{x^3} = 1$, and $x^3 + \frac{1}{x^3} = 1 - 3 = -2$.

Correct answer: A

21. The larger angles of a rhombus are double the size of the smaller angles. The shorter diagonal measures 10 inches. Find the length of a side of this rhombus.

A. 5 B. 7 C. 8 D. 10 E. 12

Let θ be the degree measure of the small angles of the rhombus. We have $2\theta + 2(2\theta) = 2\cdot 180^\circ$ and $\theta = 60^\circ$. The sides of the rhombus have the same length as the shorter diagonal.



Correct answer: D

22. Find the maximum value of the function $f(x)=\cos\left(x+\frac{\pi}{3}\right)+\cos x$.

A. 1 B. $\sqrt{2}$ C. $1/2$ D. 2 E. $\sqrt{3}$

$$f(x)=\cos\left(x+\frac{\pi}{3}\right)+\cos x=2\cos\left(\frac{x+\frac{\pi}{3}+x}{2}\right)\cos\left(\frac{x+\frac{\pi}{3}-x}{2}\right)$$

$$=2\cos\left(x+\frac{\pi}{6}\right)\cos\frac{\pi}{6}=2\cos\left(x+\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}\leq\sqrt{3}$$

Correct answer: E

23. A postal employee delivered mail daily for 42 days, each day delivering 4 more letters than on the previous day. The total delivery for the first 24 days of the period was the same as that for the last 18 days. How many letters did the employee deliver during the whole 42-day period?

A. 1000 B. 11120 C. 12096 D. 13028 E. 21434

The numbers of letters delivered daily form an arithmetic sequence with common difference $d=4$. Let a_1 be the number of letters delivered on the first day. The number of total delivery during the whole 42-day period is $S_{42}=42a_1+\frac{42(42-1)}{2}\cdot 4=42a_1+3444$. The number of total delivery for the first 24 days is $S_{24}=24a_1+\frac{24(24-1)}{2}\cdot 4=24a_1+1104$ We have $2(24a_1+1104)=42a_1+3444$, $48a_1+2208=42a_1+3444$, $6a_1=1236$, and $a_1=206$. Therefore, $S_{42}=42(206)+3444=12096$.

Correct answer: C

24. Find the maximum value of the function
$f(x)=x^2-(x-2)^2-(x-1)^2-(x-5)^2-(x-4)^2+(x-6)^2$.

A. 3 B. 8 C. 0 D. 5 E. 20

$$f(x)=x^2-(x-2)^2-(x-1)^2-(x-5)^2-(x-4)^2+(x-6)^2$$
$$=x^2-x^2+4x-4-x^2+2x-1-x^2+10x-25-x^2+8x-16+x^2-12x+36$$
$$=-2x^2+12x-10=-2(x^2-6x+9)+8=-2(x-3)^2+8$$

When $x=3$, $f(x)$ has the maximum value 8.

Correct answer: B

25. If $x = 3\sin\theta$ and $0^\circ \le \theta \le 90^\circ$, write the expression $\dfrac{\sin 2\theta}{4}$ in terms of just *x*.

A. $\dfrac{x\sqrt{9-x^2}}{18}$ B. $\dfrac{x\sqrt{9-x^2}}{9}$ C. $\dfrac{x}{12}$ D. $\dfrac{x}{6}$ E. $\dfrac{2x\sqrt{9-x^2}}{9}$

$\sin\theta = \dfrac{x}{3}$ and $0^\circ \le \theta \le 90^\circ$. $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\dfrac{x}{3}\right)^2} = \dfrac{\sqrt{9-x^2}}{3}$.

$$\frac{\sin 2\theta}{4} = \frac{2\sin\theta\cos\theta}{4} = \frac{1}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} = \frac{x\sqrt{9-x^2}}{18}$$

Correct answer: A

26. If the distance from the point (m^2, m) in Quadrant I to the line $y = x - 2$ is $2\sqrt{2}$. What is the value of m?

A. 3 B. 2 C. 1 D. -1 E. -2

The distance from (m^2, m) to the line $x - y - 2 = 0$ is $d = \dfrac{|m^2 - m - 2|}{\sqrt{2}} = 2\sqrt{2}$.

We have $|m^2 - m - 2| = 4$. $m^2 - m - 2 = -4$ or $m^2 - m - 2 = 4$.

$m^2 - m + 2 = 0$ has no real solution. $m^2 - m - 6 = 0$, $(m-3)(m+2) = 0$, $m = 3$ or $m = -2$.

Since (m^2, m) is in Quadrant I, $m = 3$ only.

Correct answer: A

27. The diagonals of a quadrilateral are 10 and 8. Find the perimeter of a new quadrilateral formed by joining the midpoints of the sides of the original quadrilateral.

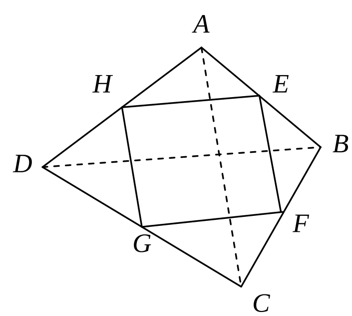
A. 36 B. 24 C. 20 D. 18 E. 9

Let *E*, *F*, *G*, and *H* be the midpoints of *AB*, *BC*, *CD*, and *DA* respectively. $AC = 10$ and $BD = 8$.

$EF = GH = \dfrac{1}{2}AC = \dfrac{1}{2}\cdot 10 = 5$

$FG = HE = \dfrac{1}{2}BD = \dfrac{1}{2}\cdot 8 = 4$

$EF + FG + GH + HE = 5 + 4 + 5 + 4 = 18$

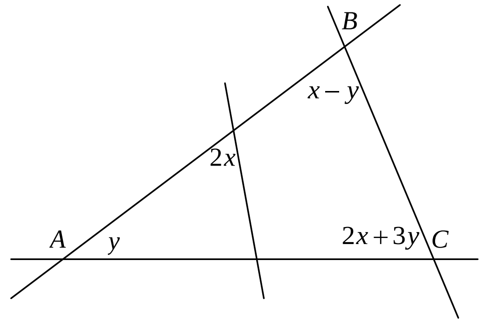


Correct answer: D

28. Given the figure, find the value of $x+y$

A. 60° B. 90° C. 120° D. 180° E. 360°

In the triangle *ABC*,
$y+(x-y)+(2x+3y)=180^\circ$, $3x+3y=180^\circ$,
and $x+y=60^\circ$.



Correct answer: A

29. An auto insurance company has 10,000 policyholders. Each policyholder is classified as
(i) young or old,
(ii) male or female, and
(iii) married or single.
Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

A. 280 B. 423 C. 486 D. 880 E. 896

3000 are young and 1320 are young males. The number of young female policyholders is $3000-1320=1680$. 1400 are young and married and 600 are young married male. There are $1400-600=800$ young married females policyholders.
The number of young, female, and single policyholders is $1680-800=880$.

Correct answer: D

30. A group of roosters want to buy an alarm clock. If each contributes \$0.35, they lack \$4.40. If each contributes \$0.40, they have \$4.40 extra. The number of roosters is in the range of

A. Less than 50 B. 50 to 100 C. 100 to 150 D. 150 to 200 E. Greater than 200

Let *x* be the number of roosters and *y* the price of the alarm clock. We have the system of equations.

$$\begin{cases} 0.35x+4.40=y \\ 0.40x-4.40=y \end{cases}$$

$0.35x+4.40=0.40x-4.40$, $0.05x=8.80$, and $x=176$.

Correct answer: D

31. (Tie Break No. 3) Find the value of $\sin 20^\circ \cos 70^\circ + \sin 10^\circ \sin 50^\circ$.

A. $\frac{1}{4}$ B. $\frac{\sqrt{3}}{2}$ C. $\frac{1}{2}$ D. $\frac{3}{4}$ E. None of the above

$$\sin 20^\circ \cos 70^\circ + \sin 10^\circ \sin 50^\circ = \frac{1}{2}\left(\sin\left(20^\circ + 70^\circ\right) + \sin\left(20^\circ - 70^\circ\right)\right) + \frac{1}{2}\left(\cos\left(10^\circ - 50^\circ\right) - \cos\left(10^\circ + 50^\circ\right)\right)$$

$$= \frac{1}{2}\left(\sin 90^\circ + \sin(-50^\circ) + \cos(-40^\circ) - \cos 60^\circ\right) = \frac{1}{2}\left(1 - \sin 50^\circ + \cos 40^\circ - \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \sin 50^\circ + \sin\left(90^\circ - 40^\circ\right)\right) = \frac{1}{2}\left(\frac{1}{2} - \sin 50^\circ + \sin 50^\circ\right) = \frac{1}{4}$$

Correct answer: A

32. Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and define $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, $\cdots\cdots$. Then $f^{99}(1) = ?$

A. 1/4 B. $1/\sqrt{99}$ C. 1/9 D. 1/10 E. None of the above

$f(1) = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$, $f^2(1) = f\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\sqrt{1+\left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{\sqrt{2}}} = \frac{1}{\sqrt{3}}$, $\cdots\cdots$. Assume that when $n = k$,

$$f^k(1) = \frac{1}{\sqrt{k+1}},\quad f^{k+1}(1) = f\left(f^k(1)\right) = f\left(\frac{1}{\sqrt{k+1}}\right) = \frac{\frac{1}{\sqrt{k+1}}}{\sqrt{1+\left(\frac{1}{\sqrt{k+1}}\right)^2}} = \frac{\frac{1}{\sqrt{k+1}}}{\sqrt{1+\frac{1}{k+1}}} = \frac{\frac{1}{\sqrt{k+1}}}{\frac{\sqrt{k+2}}{\sqrt{k+1}}} = \frac{1}{\sqrt{k+2}}$$

Therefore, we have $f^n(1) = \frac{1}{\sqrt{n+1}}$. Particularly, $f^{99}(1) = \frac{1}{\sqrt{99+1}} = \frac{1}{10}$.

Correct answer: D

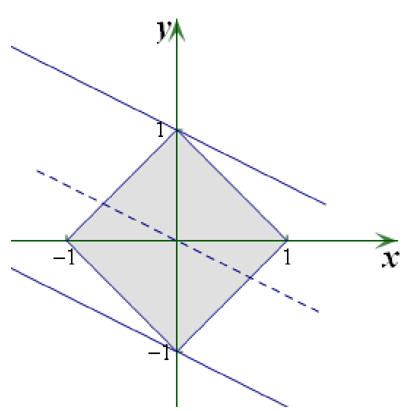
33. Assume that variables x and y satisfy $|x| + |y| \le 1$. What is the maximum value and the minimum value of $x + 2y$, respectively?

A. 1 and -1 B. $3/2$ and $-3/2$ C. 1 and -2 D. 2 and -2 E. 3 and -3

The solution set of $|x| + |y| \le 1$ shows on right side. $f(x, y) = x + 2y$ takes maximum value and minimum value at the vertices of the region.

Maximum: $f(0,1) = 0 + 2(1) = 2$

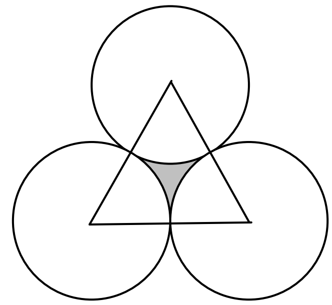
Minimum: $f(0,-1) = 0 + 2(-1) = -2$



Correct answer: D

34. In the figure below, three circles are tangent to each other. Find the area of the shaded region if each circle has a radius of 6.

A. 18 B. $18-4\pi$ C. $36\sqrt{3}$ D. $36\sqrt{3}-18\pi$ E. $36\sqrt{3}-12\pi$



Connecting the centers of the circles, we have an equilateral triangle.
The sides of the equilateral triangle have length 12 .

The area of the triangle is $A_1=\frac{1}{2}(12)^2\sin 60^\circ=36\sqrt{3}$.

The area of the total three congruent sectors is

$A_2=3\left(\frac{1}{2}\cdot 6^2\cdot\frac{\pi}{3}\right)=18\pi$.

The area of the shaded region is $A_1-A_2=36\sqrt{3}-18\pi$.

Correct answer: D

35. If $\tan 2A=2$ and A terminates in the second quadrant, find the value of $\sin A$.

A. $\frac{\sqrt{5}-1}{4}$ B. $\frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}$ C. $\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}}$ D. $\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}$ E. $\frac{\sqrt{6}-\sqrt{2}}{4}$

$\tan 2A=\frac{2\tan A}{1-\tan^2 A}=2$, $\tan^2 A+\tan A-1=0$, and $\tan A=\frac{-1\pm\sqrt{5}}{2}$. Since A is terminated in the second quadrant, $\tan A=\frac{1+\sqrt{5}}{-2}<0$ and $\sin A>0$. $\left(-2,1+\sqrt{5}\right)$ is a point on the terminal side of A.

$\sin A=\frac{1+\sqrt{5}}{\sqrt{\left(1+\sqrt{5}\right)^2+(-2)^2}}=\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}$

Correct answer: D

36. Find the value of x if $\log_4\left(\log_{25}\left(\log_3 x\right)\right)=-\frac{1}{2}$.

A. $\frac{1}{243}$ B. 5 C. 243 D. 25 E. 81

$\log_{25}\left(\log_3 x\right)=4^{-\frac{1}{2}}=\frac{1}{4^{\frac{1}{2}}}=\frac{1}{2}$, $\log_3 x=25^{\frac{1}{2}}=5$, and $x=3^5=243$.

Correct answer: C

37. From a point five nautical miles due west of a lighthouse, a ship heads due north at a constant speed of 10 knots (nautical miles per hour). How fast in knots is the ship moving away from this lighthouse one hour later?

A. $3\sqrt{5}$ B. $4\sqrt{5}$ C. $2\sqrt{7}$ D. $3\sqrt{7}$ E. 9

At the time t, the distance between the ship and lighthouse is $s=\sqrt{5^2+(10t)^2}$.

$$\frac{ds}{dt}=\frac{1}{2}\left(25+100t^2\right)^{-\frac{1}{2}}200t=\frac{100t}{\sqrt{25+100t^2}}$$

$$\left.\frac{ds}{dt}\right|_{t=1}=\frac{100\cdot 1}{\sqrt{25+100\cdot 1^2}}=\frac{100}{\sqrt{125}}=\frac{100}{5\sqrt{5}}=\frac{20}{\sqrt{5}}=\frac{20\sqrt{5}}{5}=4\sqrt{5}$$

Correct answer: B

38. (Tie Break No. 4) What is the coefficient of $x^3y^2z^3$ in the expansion of $(x-y-2z)^8$?

A. 8 B. -256 C. 560 D. -560 E. -4480

Write $(x-y-2z)^8=((x-y)-2z)^8$. Then the term of $x^3y^2z^3$ is a term in the expansion of $\binom{8}{3}(x-y)^{8-3}(-2z)^3=-488(x-y)^5z^3$. The term of x^3y^2 in the expansion of $(x-y)^5$ is $\binom{5}{2}x^{5-2}(-y)^2=10x^3y^2$. Therefore, the coefficient of $x^3y^2z^3$ in the expansion of $(x-y-2z)^8$ is $-488\cdot 10=-4880$.

Correct answer: E

39. If $\cot\varphi=-\frac{4}{3}$ and $\sin\varphi$ is negative, which of the following expressions has the least value?

A. $\cos\left(\frac{\varphi}{2}\right)$ B. $\tan\left(\varphi+\frac{\pi}{4}\right)$ C. $\cos\left(\varphi+\frac{\pi}{6}\right)$ D. $\tan\left(\frac{\varphi}{2}\right)$ E. $\csc\left(\varphi+\frac{3\pi}{2}\right)$

$\cot\varphi=-\frac{4}{3}$ and $\sin\varphi<0$ imply $\sin\varphi=\frac{-3}{\sqrt{(-3)^2+4^2}}=-\frac{3}{5}$, $\cos\varphi=\frac{4}{\sqrt{(-3)^2+4^2}}=\frac{4}{5}$, and $\tan\varphi=-\frac{3}{4}$.

$$\cos\left(\frac{\varphi}{2}\right)\ge -1;\ \tan\left(\varphi+\frac{\pi}{4}\right)=\frac{\tan\varphi+\tan\frac{\pi}{4}}{1-\tan\varphi\tan\frac{\pi}{4}}=\frac{-\frac{3}{4}+1}{1-\left(-\frac{3}{4}\right)\cdot 1}=\frac{1}{7}>-1;\ \cos\left(\varphi+\frac{\pi}{6}\right)\ge -1;$$

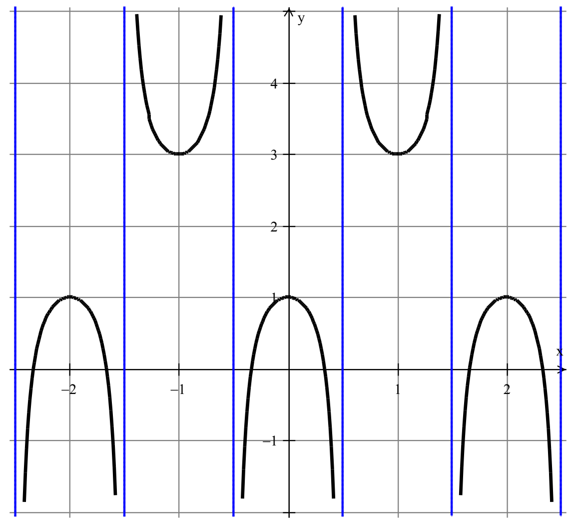
$$\tan\left(\frac{\varphi}{2}\right)=\frac{1-\cos\varphi}{\sin\varphi}=\frac{1-\frac{4}{5}}{-\frac{3}{5}}=-\frac{1}{3}>-1;\ \csc\left(\varphi+\frac{3\pi}{2}\right)=\frac{1}{\sin\left(\varphi+\frac{3\pi}{2}\right)}=\frac{1}{-\cos\varphi}=\frac{1}{-\frac{4}{5}}=-\frac{5}{4}<-1.$$

Correct answer: E

40. Which of the following functions is represented by the graph below?

A. $f(x) = 2 - \sec x$ B. $g(x) = 2 + \sec(\pi x)$ C. $h(x) = 2 - \csc\left(\pi x + \frac{\pi}{2}\right)$

D. $j(x) = 2 - \csc\left(\pi x + \frac{1}{2}\right)$ E. $k(x) = 2 - \csc\left(x + \frac{\pi}{2}\right)$



The period of the function represented by the graph is 2. $f(x)$ and $k(x)$ are excluded since their periods are 2π. The point $(0,1)$ is on the graph of the function.

$g(0) = 2 + \sec(0) = 2 + 1 = 3 \neq 1$

$j(0) = 2 - \csc\left(0 + \frac{1}{2}\right) = 2 - \csc\frac{1}{2} \neq 1$

$h(0) = 2 - \csc\left(0 + \frac{\pi}{2}\right) = 2 - 1 = 1$

Only the choice is C.

Correct answer: C

41. (Tie Break No. 5) The area of triangle $ABC = (BC)^2 + (AC)^2 - (AB)^2$. If angle C is acute, compute the numerical value of its secant.

A. $\sqrt{17}$ B. $2\sqrt{3}$ C. $\sqrt{21}$ D. $3\sqrt{5}$ E. 5

The area of the triangle ABC is also equal to $\frac{1}{2}(BC)(AC)\sin C$.

We have $\frac{1}{2}(BC)(AC)\sin C = (BC)^2 + (AC)^2 - (AB)^2$, $\frac{1}{2}\sin C = 2\frac{(BC)^2 + (AC)^2 - (AB)^2}{2(BC)(AC)} = 2\cos C$, and $\tan C = 4$. Since $0^\circ < C < 90^\circ$, $\sec C = \sqrt{1 + \tan^2 C} = \sqrt{1 + 4^2} = \sqrt{17}$.

Correct answer: A

42. Find the solution set of the equation $\log_4|2x+2| - \log_4|3x+1| = \frac{1}{2}$.

A. $\{-1/2, 0\}$ B. $\{0\}$ C. $\{1/2, 0\}$ D. $\{1, -1/2\}$ E. $\{-1/2, 1\}$

$\log_4\left|\frac{2x+2}{3x+1}\right| = \frac{1}{2}$, $\left|\frac{2x+2}{3x+1}\right| = 4^{\frac{1}{2}} = 2$, $\frac{2x+2}{3x+1} = 2$ or $\frac{2x+2}{3x+1} = -2$.

$2x + 2 = 6x + 2$, $4x = 0$, $x = 0$; or $2x + 2 = -6x - 2$, $8x = -4$, $x = -\frac{1}{2}$.

Correct answer: A

43. If $2\tan\left[\cos^{-1}\left(\frac{3}{5}\right)\right] = x$, then x = ?

A. $\frac{3}{4}$ B. $\frac{8}{3}$ C. $\frac{3}{2}$ D. $\frac{3}{8}$ E. $\frac{4}{3}$

Let $\theta = \cos^{-1}\left(\frac{3}{5}\right)$. Then $0^\circ < \theta < 90^\circ$, $\cos\theta = \frac{3}{5}$, and $\sec\theta = \frac{5}{3}$

$$x = 2\tan\left[\cos^{-1}\left(\frac{3}{5}\right)\right] = 2\tan\theta = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\left(\frac{5}{3}\right)^2 - 1} = 2\sqrt{\frac{16}{9}} = 2\cdot\frac{4}{3} = \frac{8}{3}$$

Correct answer: B

44. Find the solution set of the system $\begin{cases} x^2 + y^2 = 2a^2 - 2ab + b^2 & (1) \\ 2x^2 - y^2 = a^2 + 2ab - b^2 & (2) \end{cases}$ in terms of *a* and *b*.

A. $\{(a,\ a-b),\ (-a,\ b-a)\}$ B. $\{(a,\ b-a),\ (-a,\ a-b)\}$ C. $\{(a,\ a-b)\}$

D. $\{(a,\ a-b),\ (a,\ b-a),\ (-a,\ a-b),\ (-a,\ b-a)\}$ E. None of the above

Adding (1) and (2), $3x^2 = 3a^2$. $x = a$ or $x = -a$.
Substitute $x = a$ into (1)

$a^2 + y^2 = 2a^2 - 2ab + b^2$

$y^2 = a^2 - 2ab + b^2 = (a-b)^2$, $y = a - b$ or $y = -(a-b) = b - a$.

Similarly, substitute $x = -a$ into (1) we have $y = a - b$ or $y = -(a-b) = b - a$.

The solution set of the system is $\{(a,\ a-b),\ (a,\ b-a),\ (-a,\ a-b),\ (-a,\ b-a)\}$.

Correct answer: D

45. Suppose the function $f(x)$ satisfies $f(5) = 3$ and $f(x) = f(x-1) + 3x$. Then $f(3) + f(6) = ?$

A. 0 B. 3 C. -3 D. 6 E. 9

$f(x) = f(x-1) + 3x$, $f(6) = f(6-1) + 3\cdot 6 = f(5) + 18 = 3 + 18 = 21$

$f(x-1) = f(x) - 3x$, $f(4) = f(5-1) = f(5) - 3\cdot 5 = 3 - 15 = -12$,

$f(3) = f(4-1) = f(4) - 3\cdot 4 = -12 - 12 = -24$

$f(3) + f(6) = -24 + 21 = -3$

Correct answer: C