1. Four numbers are written in a row. The average of the first two numbers is 5. The average of the middle two numbers is 4, and the average of the last two numbers is 10. What is the average of the first and last numbers?

A. 9 B. 11 C. 10 D. 10.5 E. 11.5

Let x, y, z, w be the row of the four numbers. We have

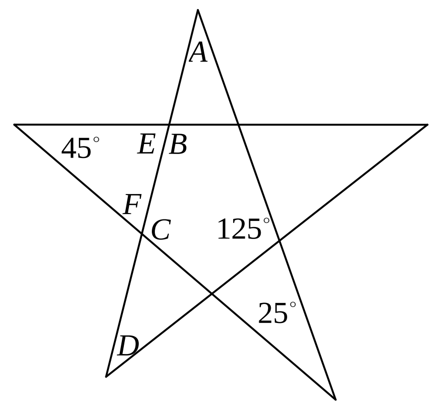
$$\begin{cases} \dfrac{x+y}{2}=5 & (1) \\ \dfrac{y+z}{2}=4 & (2) \\ \dfrac{z+w}{2}=10 & (3) \end{cases}$$

. Subtracting (2) from (1), we have $\frac{x}{2}-\frac{z}{2}=1$ (4). Adding (4) and (3) we have $\frac{x}{2}+\frac{w}{2}=11$.

Correct Answer: B

2. Using the following figure, determine the value of $m\angle A+m\angle B+m\angle C+m\angle D$.

A. 220° B. 240° C. 265° D. 280° E. 290°



$m\angle E+m\angle F=180^\circ-45^\circ=135^\circ$ (1)

$m\angle E+m\angle B+m\angle F+m\angle C=360^\circ$ (2)

Subtracting (1) from (2), we have

$m\angle B+m\angle C=225^\circ$ (3)

We also have

$m\angle A+m\angle D=180^\circ-125^\circ=55^\circ$ (4)

Adding (3) and (4), we have

$m\angle A+m\angle B+m\angle C+m\angle D=280^\circ$

Correct Answer: D

3. Find the value of $x\sin x$ if $x=\dfrac{\pi}{6}$.

A. $\pi/12$ B. $\pi\sqrt{3}/12$ C. $\pi\sqrt{2}/12$ D. $\pi/6$ E. None of the above

$$\frac{\pi}{6}\sin\frac{\pi}{6}=\frac{\pi}{6}\cdot\frac{1}{2}=\frac{\pi}{12}$$

Correct Answer: A

4. The Body Mass Index (BMI) varies directly with a person's weight (pounds) and inversely with the square of the person's height (inches). Given that a 6-foot tall man weighing 180 pounds has a BMI of 24.0, what is the BMI of a woman weighing 120 pounds with a height of 5 feet 4 inches?

A. 20.25 B. 21.50 C. 19.25 D. 23.00 E. 19.85

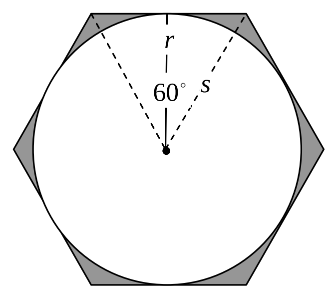
Let w and h denote the weight and height respectively. $\text{BMI} = k\dfrac{w}{h^2}$

We have $24 = k\dfrac{180}{(6\times 12)^2}$, $k = \dfrac{72^2\times 24}{180} = 691.2$ The BMI of the woman is $691.2\times\dfrac{120}{(5\times 12+4)^2} = 20.25$

Correct Answer: A

5. A circle of circumference 18π is inscribed in a regular hexagon as shown below. Find the shaded area.

A. $81(3\sqrt{3}-\pi)$ B. $27(8\sqrt{3}-3\pi)$ C. $81(2\sqrt{3}-\pi)$ D. $27(4\sqrt{3}-2\pi)$ E. $9(4\sqrt{3}-2\pi)$



The radius of the circle is $r = \dfrac{18\pi}{2\pi} = 9$ and the area of the circle is

$A_c = \pi\cdot 9^2 = 81\pi$. The length of each side of the hexagon is

$s = \dfrac{r}{\cos 30^\circ} = 9\cdot\dfrac{2}{\sqrt{3}} = 6\sqrt{3}$. The area of the hexagon is

$A_h = 6\cdot\left(\dfrac{1}{2}sr\right) = 6\cdot\left(\dfrac{1}{2}\cdot 6\sqrt{3}\cdot 9\right) = 162\sqrt{3}$. The shaded area is

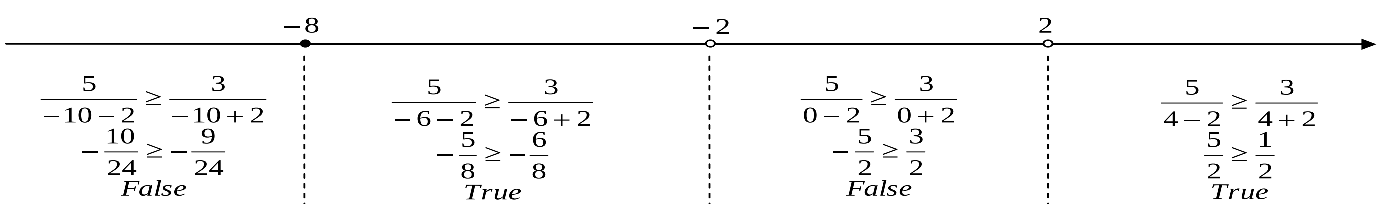
$A_h - A_c = 162\sqrt{3} - 81\pi = 81(2\sqrt{3}-\pi)$

Correct Answer: C

6. (Tie Break No.1) Solve the rational inequality: $\dfrac{5}{x-2} \ge \dfrac{3}{x+2}$

A. $(-\infty,-6]\cup(2,\infty)$ B. $[-6,-2)\cup(2,\infty)$ C. $[-8,-2)\cup(2,\infty)$ D. $(-\infty,-8)\cup(2,\infty)$ E. $(-8,-2)\cup(2,\infty)$

$\dfrac{5}{x-2} - \dfrac{3}{x+2} \ge 0$, $\dfrac{5(x+2)-3(x-2)}{(x-2)(x+2)} \ge 0$, and $\dfrac{2(x+8)}{(x-2)(x+2)} \ge 0$. The zero of the numerator is $x = -8$ (inclusive) and the zeros of the denominator are $x = -2$ and $x = 2$ (exclusive). They divide the number line into four intervals. In each interval, pick a point to check the inequality. It is true on $[-8,-2)\cup(2,\infty)$.



Correct Answer: C

7. Let $\{a_n\}$ be an arithmetic sequence and $\{s_n\}$ be its partial sum sequence. If $\lim_{n\to\infty}\frac{s_n}{n^2}=4$, find the common difference of $\{a_n\}$.

A. 4 B. 2 C. 6 D. 8 E. Not enough information

$$s_n = na_1 + \frac{n(n-1)}{2}d,\ \lim_{n\to\infty}\frac{s_n}{n^2} = \lim_{n\to\infty}\left(\frac{a_1}{n}+\frac{n-1}{2n}d\right) = \frac{1}{2}d = 4,\ d = 8$$

Correct Answer: D

8. Find the value of $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

A. 0 B. 16 C. 27 D. 81 E. $3\sqrt{3}$

By base change formula, the expression is equivalent to $\log_2\left(\frac{\log_2 x}{\log_2 8}\right) = \frac{\log_2(\log_2 x)}{\log_2 8}$,

$\log_2\left(\frac{\log_2 x}{3}\right) = \frac{\log_2(\log_2 x)}{3}$, $\log_2(\log_2 x) - \log_2 3 = \frac{1}{3}\log_2(\log_2 x)$, $\frac{2}{3}\log_2(\log_2 x) = \log_2 3$,

$2\log_2(\log_2 x) = 3\log_2 3$, and $\log_2(\log_2 x)^2 = \log_2 3^3$. Therefore, $(\log_2 x)^2 = 3^3 = 27$.

Correct Answer: C

9. Solve for x, given $\begin{vmatrix} x+a & x & x \\ x & x+b & x \\ x & x & x+c \end{vmatrix} = 0$

A. $x = a+b+c$ B. $x = \frac{1}{a+b+c}$ C. $x = -\frac{abc}{ab+bc+ca}$ D. $x = \frac{ab+bc+ca}{abc}$ E. $x = \frac{1}{ab+bc+ca}$

$$\begin{vmatrix} x+a & x & x \\ x & x+b & x \\ x & x & x+c \end{vmatrix} = \begin{vmatrix} 1 & x & x & x \\ 0 & x+a & x & x \\ 0 & x & x+b & x \\ 0 & x & x & x+c \end{vmatrix} \overset{\substack{R_2=r_2-r_1 \\ R_3=r_3-r_1 \\ R_4=r_4-r_1}}{=} \begin{vmatrix} 1 & x & x & x \\ -1 & a & 0 & 0 \\ -1 & 0 & b & 0 \\ -1 & 0 & 0 & c \end{vmatrix} = 0$$

Expanding the determinant across the first row, we have

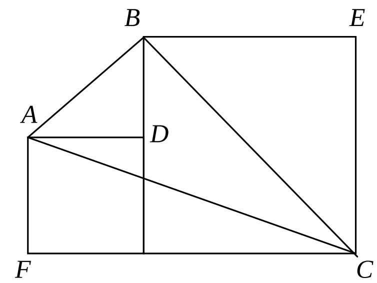
$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} - x\begin{vmatrix} -1 & 0 & 0 \\ -1 & b & 0 \\ -1 & 0 & c \end{vmatrix} + x\begin{vmatrix} -1 & a & 0 \\ -1 & 0 & 0 \\ -1 & 0 & c \end{vmatrix} - x\begin{vmatrix} -1 & a & 0 \\ -1 & 0 & b \\ -1 & 0 & 0 \end{vmatrix} = abc + xbc + xca + xab = 0.$$

Therefore, $x = -\frac{abc}{ab+bc+ca}$

Correct Answer: C

10. In the figure, the length of sides of the larger square is *a*. The length of sides of the smaller square is *b*. Find the area of the triangle *ABC*.

A. $\dfrac{ab}{2(a+b)}$ B. $\dfrac{b^2}{2(a+b)}$ C. $\dfrac{a^2}{2(a+b)}$ D. $\dfrac{1}{2}a^2$ E. None of the above



The area of ΔABD is $\frac{1}{2}(a-b)b$. The area of the pentagon $ABECF$ is $A_1 = a^2 + b^2 + \frac{1}{2}(a-b)b$. The area of ΔAFC is $A_2 = \frac{1}{2}b(a+b)$ and the area of ΔBEC is $A_3 = \frac{1}{2}a^2$. Then, the area of ΔABC is equal to

$$A_1 - A_2 - A_3 = a^2 + b^2 + \frac{1}{2}(a-b)b - \frac{1}{2}b(a+b) - \frac{1}{2}a^2 = \frac{1}{2}a^2$$

Correct Answer: D

11. Find the solution set of the equation $|3x+2| + |3x-5| = 7$

A. $\{-2/3,\ 5/3\}$ B. $[-2/3,\ 5/3]$ C. $(-2/3,\ 5/3)$ D. $(-2/3, 5/3]$ E. None of the above

$|3x+2| + |3x-5| \ge |3x+2-(3x-5)| = 7$. The equality holds if and only if $(3x+2)(3x-5) \le 0$, i.e. $-\frac{2}{3} \le x \le \frac{5}{3}$. The solution set of the equation is $[-2/3,\ 5/3]$.

Correct Answer: B

12. How many subsets does the set of all the rational solutions of the equation $3x^4 - x^3 - 11x^2 + 3x + 6 = 0$ have?

A. 4 B. 2 C. 1 D. 8 E. 16

Because the sum of all the coefficients of $3x^4 - x^3 - 11x^2 + 3x + 6$ is 0, $x = 1$ is a rational solution of the equation. By synthetic division, $3x^4 - x^3 - 11x^2 + 3x + 6 = (x-1)(3x^3 + 2x^2 - 9x - 6)$. Factoring out by grouping, $3x^3 + 2x^2 - 9x - 6 = (3x+2)(x^2-3)$. The equation has two rational solutions, $\left\{1, -\frac{2}{3}\right\}$. The set $\left\{1, -\frac{2}{3}\right\}$ has $2^2 = 4$ subsets.

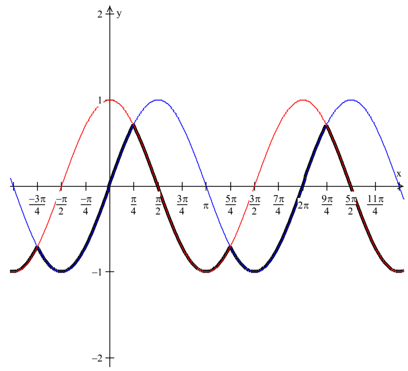
Correct Answer: A

13. Find the range of the function $f(x)=\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}|\sin x-\cos x|$

A. $[-\sqrt{2}/2, 1]$ B. $[-1, \sqrt{2}/2]$ C. $[-1, 1]$ D. $[-1, \sqrt{3}/2]$ E. $[-\sqrt{3}/2, 1]$

$$f(x)=\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}|\sin x-\cos x|=\begin{cases}\cos x & \text{if } \sin x\ge\cos x\\ \sin x & \text{if } \cos x>\sin x\end{cases}$$

That means $f(x)=\min\{\sin x, \cos x\}$
Its graph is given on the right side.
Therefore, its range is $[-1, \sqrt{2}/2]$



Correct Answer: B

14. (Tie Break No.2) How many positive integer solutions does the equation $x+y+z+w=15$ have?

A. 728 B. 182 C. 324 D. 546 E. 364

Let us place 15 small balls in a line. Then insert 3 chips between the balls and separate balls into 4 groups. Each such separation corresponds to one and only one positive integer solution of the equation. There are 14 slots between 15 balls. We have ${}_{14}C_3=364$ ways to insert 3 chips.

Correct Answer: E

15. Find the maximum value of the function $f(x)=\sqrt{x+2}+\sqrt{8-2x}$.

A. $3\sqrt{2}$ B. $3\sqrt{3}$ C. $\sqrt{3}+\sqrt{6}$ D. $\sqrt{5}+\sqrt{2}$ E. None of the above

Method 1: $f'(x)=\frac{1}{2\sqrt{x+2}}+\frac{-2}{2\sqrt{8-2x}}=\frac{\sqrt{8-2x}-2\sqrt{x+2}}{2\sqrt{(x+2)(8-2x)}}$. Solve $\sqrt{8-2x}=2\sqrt{x+2}$, $8-2x=4(x+2)$, $x=0$. When $-2<x<0$, $f'(x)>0$, $f(x)\uparrow$ and when $0<x<4$, $f'(x)<0$, $f(x)\downarrow$. Therefore, $f(x)$ has the maximum value $f(0)=3\sqrt{2}$ at $x=0$.

Method 2: Let $a, b, c,$ and d be four positive real numbers. We have (Cauchy-Schwartz inequality)
$$(a^2+b^2)(c^2+d^2)=a^2c^2+a^2d^2+b^2c^2+b^2d^2\ge a^2c^2+2(ad)(bc)+b^2d^2=(ac+bd)^2$$
The equality holds if and only if $ad=bc$. Applying the inequality we proved, we have
$$(f(x))^2=\left(1\cdot\sqrt{x+2}+\sqrt{2}\cdot\sqrt{4-x}\right)^2\le\left(1^2+\left(\sqrt{2}\right)^2\right)\left(\left(\sqrt{x+2}\right)^2+\left(\sqrt{4-x}\right)^2\right)=3\cdot 6=18=\left(3\sqrt{2}\right)^2$$
When $1\cdot\sqrt{4-x}=\sqrt{2}\cdot\sqrt{x+2}$ that implies $x=0$, the function has the maximum value $f(0)=3\sqrt{2}$.

Correct Answer: A

16. Find the value of $\sin 15^\circ$.

A. $\frac{\sqrt{2}-\sqrt{6}}{4}$ B. $\frac{\sqrt{2}+\sqrt{6}}{4}$ C. $\frac{\sqrt{6}-\sqrt{2}}{4}$ D. $\frac{\sqrt{6}-\sqrt{2}}{2}$ E. $\frac{1}{4}$

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Correct Answer: C

17. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = ?$

A. $1/2$ B. $2/3$ C. $1/4$ D. $1/3$ E. $2/5$

Let $\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$. $A(n+1)(n+2) + Bn(n+2) + Cn(n+1) = 1$. Let $n = 0$, $A = \frac{1}{2}$; let $n = -1$, $B = -1$; let $n = -2$, $C = \frac{1}{2}$.

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} = \lim_{N\to\infty}\left(\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1} + \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n+2}\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \left(-1 + \sum_{n=1}^{N}\frac{1}{n} + \frac{1}{N+1}\right) + \frac{1}{2}\left(-1 - \frac{1}{2} + \sum_{n=1}^{N}\frac{1}{n} + \frac{1}{N+1} + \frac{1}{N+2}\right)\right)$$

$$= \lim_{N\to\infty}\left(1 - \frac{1}{N+1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{2(N+1)} + \frac{1}{2(N+2)}\right) = \frac{1}{4}$$

Correct Answer: C

18. Let a, b, and c be positive integers. If $c = (a+bi)^2 - 46i$, where $i^2 = -1$, then $a+b+c = ?$

A. 624 B. 548 C. 552 D. 456 E. 349

$c = (a+bi)^2 - 46i = a^2 + 2abi - b^2 - 46i$. We have $c = a^2 - b^2$ and $2abi - 46i = 0$. $ab = 23$ Because 23 is a prime number, we have $a = 1$, $b = 23$ or $a = 23$, $b = 1$.
In the first case, $c = a^2 - b^2 = 1^2 - 23^2 < 0$. It is impossible since c is a positive integer.
In the second case, $c = 23^2 - 1^2 = 528$, $a + b + c = 23 + 1 + 528 = 552$

Correct Answer: C

19. Solve for the matrix A if $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} A + \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$.

A. $\begin{bmatrix} 15 & -11 \\ -2 & 1 \end{bmatrix}$ B. $\begin{bmatrix} -2 & -11 \\ 1 & 15 \end{bmatrix}$ C. $\begin{bmatrix} -2 & 15 \\ 1 & -11 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 15 \\ -2 & -11 \end{bmatrix}$ E. None of the above

$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 = 3r_2 - 2r_1]{R_1 = 3r_1 - 4r_2} \begin{bmatrix} 1 & 0 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{bmatrix}$. We have $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.

$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} A = \begin{bmatrix} -5 & 1 \\ -4 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -5 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -5 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ -2 & -11 \end{bmatrix}$

Correct Answer: D

20. Find the sum $8 + 5 + 2 - 1 - \cdots\cdots - 340$.

A. -15624 B. -21232 C. -19426 D. -19422 E. -18422

It is an arithmetic series with $a_1 = 8$ and $d = -3$. We have $a_n = a_1 + (n-1)d = 8 + (n-1)(-3) = 11 - 3n$

$11 - 3n = -340$, $-3n = -351$, and $n = 117$. $S_{117} = \frac{117}{2}(a_1 + a_{117}) = \frac{117}{2}(8 + (-340)) = -19422$

Correct Answer: D

21. (Tie Break No.3) Find the area of the region enclosed by $y = 0$ and a quadratic function $y = f(x)$ that has vertex $(3, 2)$ and passes through $(1, 0)$.

A. $16/3$ B. 5 C. $15/4$ D. 6 E. None of the above

The equation of the quadratic function is $f(x) = a(x-3)^2 + 2$. It passes through $(1, 0)$.

$f(1) = a(1-3)^2 + 2 = 0$. $4a + 2 = 0$ implies $a = -\frac{1}{2}$. Solve $-\frac{1}{2}(x-3)^2 + 2 = 0$. $(x-3)^2 = 4$, $x - 3 = \pm 2$, and $x = 1$ or $x = 5$. That means $f(x)$ and $y = 0$ intersect at $x = 1$ and $x = 5$.

The area of the region is $\int_1^5 \left(-\frac{1}{2}(x-3)^2 + 2 \right) dx = \left(-\frac{(x-3)^3}{6} + 2x \right) \Big|_1^5 = \frac{16}{3}$

Correct Answer: A

22. If $a^2+b^2+c^2+1=ab+\frac{3}{2}bc+c$, then $a=?$

A. $1/2$ B. $1/4$ C. 2 D. 1 E. a can be any real number

By completing the square, $a^2-ab+\frac{1}{4}b^2+\frac{3}{4}b^2-\frac{3}{2}bc+\frac{3}{4}c^2+\frac{1}{4}c^2-c+1=0$ and $\left(a-\frac{b}{2}\right)^2+\frac{3}{4}(b-c)^2+\left(\frac{c}{2}-1\right)^2=0$. We have $\frac{c}{2}-1=b-c=a-\frac{b}{2}=0$. Then, $c=2$, $b=c=2$, and $a=\frac{b}{2}=1$.

Correct Answers: D

23. Among all the pairs of numbers $(x,\ y)$ such that $2x+y=20$, find the pair for which the sum of the squares is minimum.

A. $(4,12)$ B. $(10,0)$ C. $(7,6)$ D. $(6,8)$ E. None of the above

$y=20-2x$ implies $x^2+y^2=x^2+(20-2x)^2=5x^2-80x+400$. It is a quadratic function with $a=5>0$ and has the minimum value when $x=-\frac{(-80)}{2\times 5}=8$. The corresponding y equals $20-2(8)=4$. The pair which makes the sum of the squares minimum is $(8,4)$.

Correct Answer: E

24. If $\sin\theta+\cos\theta=\frac{1}{2}$, what is the value of $\sin^3\theta+\cos^3\theta$?

A. $3/8$ B. $5/16$ C. $1/8$ D. $5/8$ E. $11/16$

$(\sin\theta+\cos\theta)^2=\frac{1}{4}$, $\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{4}$, $2\sin\theta\cos\theta+1=\frac{1}{4}$, and $\sin\theta\cos\theta=-\frac{3}{8}$.

$(\sin\theta+\cos\theta)^3=\frac{1}{8}$, $\sin^3\theta+3\sin^2\theta\cos\theta+3\sin\theta\cos^2\theta+\cos^3\theta=\frac{1}{8}$, and

$\sin^3\theta+\cos^3\theta=\frac{1}{8}-3\sin^2\theta\cos\theta-3\sin\theta\cos^2\theta=\frac{1}{8}-3\sin\theta\cos\theta(\sin\theta+\cos\theta)=\frac{1}{8}-3\cdot\left(-\frac{3}{8}\right)\cdot\frac{1}{2}=\frac{11}{16}$.

Correct Answer: E

25. A man bought some horses for \$900. After training them, he sold all but one to a dude-ranch for \$1,710, thereby realizing a gain of 100% on the horses he sold. How many horses did he buy?

A. 6 B. 8 C. 10 D. 12 E. 14

Let x be the number of horses the man bought. We have $\frac{900}{x} = \frac{1710-900}{x-1}$. $900(x-1) = 810x$, $900x - 900 = 810x$, $90x = 900$, and $x = 10$.

Correct Answer: C

26. (Tie Break No.4) Find the remaining real zero for the 4th degree polynomial function with the following properties: it only has real coefficients, $1/2$ and 1 + 2i are two of its zeros, its y-intercept is -30, and its leading coefficient is 2.

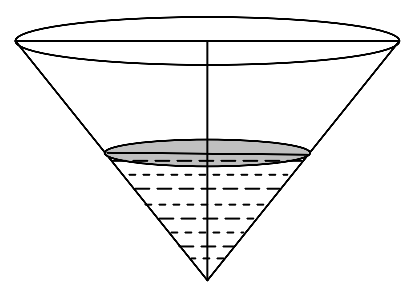
A. 8 B. 6 C. -8 D. -6 E. Cannot be determined

$x = 1-2i$ is a complex zero of the polynomial too since the polynomial has real coefficients. Let r be the remaining real zero. The polynomial has the form $p(x) = k(x-1-2i)(x-1+2i)(2x-1)(x-r)$.
$p(x) = k\left((x-1)^2 - (2i)^2\right)(2x-1)(x-r) = k(x^2 - 2x + 5)(2x-1)(x-r) = k(2x^3 - 5x^2 + 12x - 5)(x-r)$
Its leading term is $2kx^4$. $2k = 2$ and $k = 1$. Its y-intercept is $5r = -30$. Then, $r = -6$.

Correct Answer: D

27. A water tank is in the shape of an inverted right circular cone, with the height of the cone equal to the diameter of the base. If water is poured into the tank until its height reaches $1/2$ that of the tank, find the ratio of the volume of water to the volume of the tank.

A. $1/2$ B. $2/3$ C. $1/8$ D. $7/8$ E. Not enough information is provided



Let h be the height of the tank. Its radius of the base is $\frac{h}{2}$. The volume of the tank is $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi h^3}{12}$. The height of water is $\frac{h}{2}$ and the radius of the base of water is $\frac{h}{4}$. The volume of the water is $V_w = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 \cdot \frac{h}{2} = \frac{\pi h^3}{96}$. We have $\frac{V_w}{V} = \frac{\pi h^3}{96} \cdot \frac{12}{\pi h^3} = \frac{1}{8}$.

Correct Answer: C

28. Suppose the measure of an exterior angle of a regular polygon is 24°. How many sides does the polygon have?

A. 12 B. 15 C. 18 D. 20 E 24

Let n be the number of the sides of the polygon. $(n-2)180^\circ = (180^\circ - 24^\circ)n$, $180^\circ n - 360^\circ = 156^\circ n$, $24^\circ n = 360^\circ$, and $n = 15$

Correct Answer: B

29. The supplement of a certain angle divided by 3 times its complement equals $5/3$. Find the degree measure of the angle.

A. 22.5° B. 67.5° C. 112.5° D. 157.5° E. None of the above

Let θ be the degree measure of the angle. We have $\frac{180^\circ - \theta}{3(90^\circ - \theta)} = \frac{5}{3}$. $180^\circ - \theta = 5(90^\circ - \theta)$,

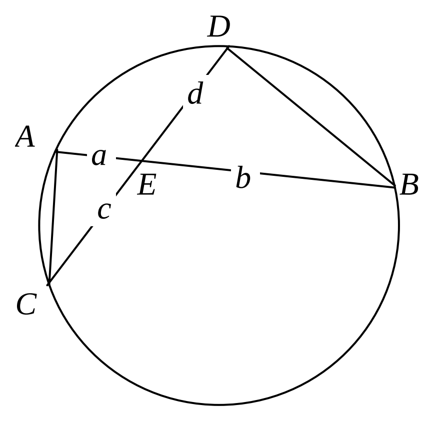
$180^\circ - \theta = 450^\circ - 5\theta$, $4\theta = 270^\circ$, and $\theta = 67.5^\circ$

Correct Answer: B

30. Two chords intersect within a circle. The segments of the first chord are 3 and 11. One segment of the second chord is 4. Find the length of the other segment of the second chord.

A. 12 B. 10 C. 8 D. 6 E. None of the above

In the figure, AB and CD are two chords intersecting at E within the circle. We have $\angle A = \angle D$ and $\angle C = \angle B$. $\Delta ACE \sim \Delta DBE$.
$\frac{c}{b} = \frac{a}{d}$ and $ab = cd$.
Let x be the length of the other segment of the second chord. By the above observation, we have $3 \cdot 11 = 4x$ and $x = \frac{33}{4}$.



Correct Answer: E

31. If a square and a circle have the same perimeter, then the ratio of the area of the square to the area of the circle is

A. $\pi/4$ B. $\pi/2$ C. $\pi^2/2$ D. $\pi^2/4$ E. None of the above

Let p be the perimeter. The area of the square is $A_s = \left(\frac{p}{4}\right)^2 = \frac{p^2}{16}$. The radius of the circle is $\frac{p}{2\pi}$ and the area of the circle is $A_c = \pi\left(\frac{p}{2\pi}\right)^2 = \frac{p^2}{4\pi}$. Then, $\frac{A_s}{A_c} = \frac{p^2}{16}\cdot\frac{4\pi}{p^2} = \frac{\pi}{4}$.

Correct Answer: A

32. Write the trigonometric expression $\sin(\sin^{-1} u - \cos^{-1} v)$ into an algebraic expression in terms of u and v. You may assume $-1 \le u \le 1$ and $-1 \le v \le 1$.

A. $uv + \sqrt{1-u^2}\sqrt{1-v^2}$ B. $uv - \sqrt{1-u^2}\sqrt{1-v^2}$ C. $u\sqrt{1-v^2} + v\sqrt{1-u^2}$ D. $u\sqrt{1-v^2} - v\sqrt{1-u^2}$ E. $u\sqrt{1-u^2} + v\sqrt{1-v^2}$

Let $A = \sin^{-1} u$. Then, $-90^\circ \le A \le 90^\circ$, $\sin A = u$, and $\cos A = \sqrt{1-u^2}$.
Let $B = \cos^{-1} v$. Then, $0^\circ \le B \le 180^\circ$, $\cos B = v$, and $\sin B = \sqrt{1-v^2}$.
$\sin(\sin^{-1} u - \cos^{-1} v) = \sin(A-B) = \sin A\cos B - \cos A\sin B = uv - \sqrt{1-u^2}\sqrt{1-v^2}$.

Correct Answer: B

33. The sum of the lengths of the twelve edges of a closed rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

A. 784 B. 776 C. 798 D. 800 E. 812

Let l, w, and h be the length, width, and height of the box respectively. $4(l+w+h) = 140$ and $\sqrt{l^2+w^2+h^2} = 21$. $l+w+h = 35$ and $l^2+w^2+h^2 = 441$. The surface area of the box is
$2lw + 2wh + 2hl = (l+w+h)^2 - (l^2+w^2+h^2) = 35^2 - 441 = 1225 - 441 = 784$

Correct Answer: A

34. Find the sum of all x-values satisfying the equation $2\cos 3x+1=0$ on the interval $[0,\ \pi]$.

A. $29\pi/18$ B. $14\pi/9$ C. $29\pi/6$ D. $14\pi/3$ E. $8\pi/3$

$\cos 3x=-\dfrac{1}{2}$. $3x=\dfrac{2\pi}{3}+2k\pi$ or $3x=\dfrac{4\pi}{3}+2k\pi$. $x=\dfrac{2\pi}{9}+\dfrac{2k\pi}{3}$ or $x=\dfrac{4\pi}{9}+\dfrac{2k\pi}{3}$. $x=\dfrac{2\pi}{9}$, $\dfrac{4\pi}{9}$, and $\dfrac{8\pi}{9}$ are the solutions on the interval $[0,\ \pi]$. Their sum is $\dfrac{14\pi}{9}$

Correct Answer: B

35. Two pulleys, one with radius 3 inches and the other with radius 10 inches, are connected by a belt. If the 3-inch pulley is caused to rotate at 3 revolutions per minute, at how many revolutions per minute does the 10-inch pulley rotate?

A. $3/20$ rpm B. $3/10$ rpm C. $9/10$ rpm D. $9/20$ rpm E. $10/3$ rpm

The circumferences of the two pulleys are 6π and 20π inches respectively. The speed of the belt is $3\cdot 6\pi=18\pi$ inches per minute. The 10-inch pulley rotates at $\dfrac{18\pi}{20\pi}=\dfrac{9}{10}$ revolutions per minute.

Correct Answer: C

36. $\tan^{-1}1+\tan^{-1}(1/2)=?$

A. $\sin^{-1}\left(1/\sqrt{10}\right)$ B. $\cos^{-1}\left(1/\sqrt{10}\right)$ C. $\cos^{-1}\left(3/\sqrt{10}\right)$ D. $\tan^{-1}(3/2)$ E. $\sin^{-1}\left(1/\sqrt{5}\right)$

Let $A=\tan^{-1}1$ and $B=\tan^{-1}\dfrac{1}{2}$. Then, $0^\circ<A+B<90^\circ$. We have $\tan A=1$, $\sin A=\dfrac{1}{\sqrt{2}}$, $\cos A=\dfrac{1}{\sqrt{2}}$,

$\tan B=\dfrac{1}{2}$, $\sin B=\dfrac{1}{\sqrt{5}}$, and $\cos B=\dfrac{2}{\sqrt{5}}$. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{1+\frac{1}{2}}{1-1\cdot\frac{1}{2}}=3$,

$\sin(A+B)=\sin A\cos B+\cos A\sin B=\dfrac{1}{\sqrt{2}}\cdot\dfrac{2}{\sqrt{5}}+\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{5}}=\dfrac{3}{\sqrt{10}}$, and

$\cos(A+B)=\cos A\cos B-\sin A\sin B=\dfrac{1}{\sqrt{2}}\cdot\dfrac{2}{\sqrt{5}}-\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{5}}=\dfrac{1}{\sqrt{10}}$

Correct Answer: B

37. (Tie Break No.5) Suppose $\sin x = \cos y$, $\sqrt{6}\sin y = \tan z$, and $2\sin z = \sqrt{3}\cos x$. Let *u*, *v*, and *w* denote $\sin^2 x$, $\sin^2 y$, and $\sin^2 z$ respectively. Find the value of the ordered triple $(u,\ v,\ w)$.

A. $(1/2,\ 1/2,\ 1/2)$ B. $(3/4,\ 1/4,\ 1/2)$ C. $(1/2,\ 1/2,\ 3/4)$ D. $(1/4,\ 3/4,\ 1/2)$ E. $(1,\ 0,\ 0)$

$\sin^2 x = \cos^2 y = 1 - \sin^2 y$, $\sin^2 y = 1 - \sin^2 x$ (1)

$4\sin^2 z = 3\cos^2 x$, $4 - 4\cos^2 z = 3 - 3\sin^2 x$, $4 + 3\sin^2 x = 3 + 4\cos^2 z$, $\cos^2 z = \dfrac{1+3\sin^2 x}{4}$ (2)

$6\sin^2 y = \tan^2 z = \sec^2 z - 1 = \dfrac{1}{\cos^2 z} - 1$, $6\cos^2 z\sin^2 y = 1 - \cos^2 z$ (3)

Substituting (1) and (2) in (3) , we have $6\left(\dfrac{1+3\sin^2 x}{4}\right)(1-\sin^2 x) = 1 - \dfrac{1+3\sin^2 x}{4}$

$6(1+3\sin^2 x)(1-\sin^2 x) = 4 - (1+3\sin^2 x)$, $6(1+2\sin^2 x - 3\sin^4 x) = 3 - 3\sin^2 x$,
$2(1+2\sin^2 x - 3\sin^4 x) = 1 - \sin^2 x$, $2 + 4\sin^2 x - 6\sin^4 x - 1 + \sin^2 x = 0$, $1 + 5\sin^2 x - 6\sin^4 x = 0$,
$(1+6\sin^2 x)(1-\sin^2 x) = 0$, $\sin^2 x = 1$, $u = 1$. Therefore, $\sin^2 y = 1 - \sin^2 x = 1 - 1 = 0$, $v = 0$,
$4\sin^2 z = 3\cos^2 x = 3(1-\sin^2 x) = 3(1-1) = 0$, $\sin^2 z = 0$, and $w = 0$.

Correct Answer: E

38. In the figure, *ABCD* is a square with side length *a*. *E* is the midpoint of *BC*. *AE* intersects the diagonal *BD* at *F*. Find the area of triangle *ABF*.

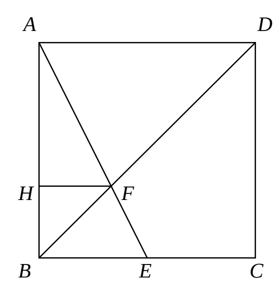
A. $a^2/5$ B. $a^2/8$ C. $a^2/6$ D. $2a^2/9$ E. None of the above

$\Delta AFD \sim \Delta EFB$. $\dfrac{FE}{AF} = \dfrac{BE}{AD} = \dfrac{1}{2}$, $AF = 2FE$, and $\dfrac{AF}{AE} = \dfrac{2}{3}$.

Through *F*, draw *HF* perpendicular to *AB*. Then $\Delta AHF \sim \Delta ABE$

$\dfrac{HF}{BE} = \dfrac{AF}{AE} = \dfrac{2}{3}$. $HF = \dfrac{2}{3}BE = \dfrac{2}{3}\cdot\dfrac{1}{2}a = \dfrac{1}{3}a$

The area of $\Delta ABF = \dfrac{1}{2}(HF)(AB) = \dfrac{1}{2}\cdot\dfrac{1}{3}a\cdot a = \dfrac{1}{6}a^2$



Correct Answer: C

39. Solve the compound inequality: $3x + 3 > 4x$ or $2(x+3) > 6$.

A. $(0, 3)$ B. $(-\infty,\ \infty)$ C. $(-\infty, 0)\cup(3,\ \infty)$ D. $(0, 3)\cup(3,\ \infty)$ E. ϕ

$3x + 3 > 4x$, $3 > x$. The solution set is $(-\infty, 3)$. $2(x+3) > 6$, $x + 3 > 3$, $x > 0$. The solution set is $(0,\ \infty)$.
Their union is $(-\infty, 3)\cup(0,\ \infty) = (-\infty,\ \infty)$.

Correct Answer: B

40. Let $f(x)=\cos^2 x,\ g(x)=\dfrac{1}{x},$ and $h(x)=\sin x$. What is $\left(f\circ\left(g^{-1}\circ h\right)^{-1}\right)(2)$?

A. $1/4$ B. $3/4$ C. 1 D. $\sqrt{3}/2$ E. 3

$g^{-1}(x)=\dfrac{1}{x}$, $\left(g^{-1}\circ h\right)(x)=\dfrac{1}{\sin x}=\csc x$, $\left(g^{-1}\circ h\right)^{-1}(x)=\csc^{-1}x$, $\left(g^{-1}\circ h\right)^{-1}(2)=\csc^{-1}2=\dfrac{\pi}{6}$, and

$\left(f\circ\left(g^{-1}\circ h\right)^{-1}\right)(2)=\cos^2\dfrac{\pi}{6}=\left(\dfrac{\sqrt{3}}{2}\right)^2=\dfrac{3}{4}$.

Correct Answer: B

41. Let both x and y be two-digit, positive integers whose mean is 80. What is the minimum possible value of their product xy ?

A. 1600 B. 6000 C. 6241 D. 6039 E. 6400

Both x and y are two-digit, positive integers. We have $10\le x\le 99$ and $10\le y\le 99$. $\dfrac{x+y}{2}=80$, $x+y=160$, and $y=160-x$. The range of x-value is the set $S=\{x\mid 61\le x\le 99,\ x \text{ is an integer}\}$.
$P(x)=xy=x(160-x)=-x^2+160x$. Its graph is a parabola with the vertex $(80, 6400)$ and opening downward. At $x=61$ or $x=99$, $P(x)$ on the set S reaches the minimum value
$61(160-61)=99(160-99)=61\cdot 99=6039$.

Correct Answer: D

42. What is the value of the following series: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n}{1-4n^2}$?

A. 0 B. $1/4$ C. $-1/4$ D. $-1/3$ E. None of the above, it diverges.

$$\sum_{n=0}^{\infty}\frac{(-1)^n n}{1-4n^2}=\lim_{N\to\infty}\sum_{n=0}^{N}(-1)^n\frac{1}{4}\left(\frac{1}{1-2n}-\frac{1}{1+2n}\right)$$

$$=\lim_{N\to\infty}\frac{1}{4}\left(-\left(-\frac{1}{1}-\frac{1}{3}\right)+\left(-\frac{1}{3}-\frac{1}{5}\right)-\left(-\frac{1}{5}-\frac{1}{7}\right)+\left(-\frac{1}{7}-\frac{1}{9}\right)-\cdots\cdots+(-1)^N\left(\frac{1}{1-2N}-\frac{1}{1+2N}\right)\right)$$

$$=\lim_{N\to\infty}\frac{1}{4}\left(1+\frac{1}{3}-\frac{1}{3}-\frac{1}{5}+\frac{1}{5}+\frac{1}{7}-\frac{1}{7}-\frac{1}{9}+\cdots+(-1)^N\frac{1}{1-2N}-(-1)^N\frac{1}{1+2N}\right)$$

$$=\lim_{N\to\infty}\frac{1}{4}\left(1-(-1)^N\frac{1}{1+2N}\right)=\frac{1}{4}$$

Correct Answer: B

43. Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$ and $y=\sqrt{6-\sqrt{6-\sqrt{6-\cdots}}}$. Find $x+y$.

A. 6 B. 0 C. $2\sqrt{6}$ D. $2\sqrt{3}$ E. 5

Let $x_n=\overbrace{\sqrt{6+\sqrt{6+\sqrt{6+\cdots+\sqrt{6}}}}}^{n\ radicals}$, $0<x_n<\sqrt{6+6}=2\sqrt{3}$, and $x_n<x_{n+1}$. $x=\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}=\lim_{n\to\infty}x_n$ exists and $x\ge 0$. Let $y_n=\overbrace{\sqrt{6-\sqrt{6-\sqrt{6-\cdots-\sqrt{6}}}}}^{n\ radicals}$, $0<y_n$, and $y_{n+1}<y_n$. $y=\sqrt{6-\sqrt{6-\sqrt{6-\cdots}}}=\lim_{n\to\infty}y_n$ exists and $y\ge 0$. $\lim_{n\to\infty}x_n=\lim_{n\to\infty}\sqrt{6+x_{n-1}}$, $x=\sqrt{6+x}$, $x^2-x-6=0$, $(x-3)(x+2)=0$, and $x=3$. $\lim_{n\to\infty}y_n=\lim_{n\to\infty}\sqrt{6-y_{n-1}}$, $y=\sqrt{6-y}$, $y^2+y-6=0$, $(y+3)(y-2)=0$, $y=2$, and $x+y=5$.

Correct Answer: E

44. If $3\sin\theta+4\cos\theta=5$, then $\tan\theta=?$

A. 1 B. -1 C. $3/4$ D. $4/3$ E. 0

$\frac{3}{5}\sin\theta+\frac{4}{5}\cos\theta=1$. Let $\cos\alpha=\frac{3}{5}$ and $\sin\alpha=\frac{4}{5}$. Then, $\tan\alpha=\frac{4}{3}$. $\sin\theta\cos\alpha+\cos\theta\sin\alpha=1$, $\sin(\theta+\alpha)=1$, $\theta+\alpha=\frac{\pi}{2}+2k\pi$ and $\theta=\frac{\pi}{2}-\alpha+2k\pi$. Therefore, $\tan\theta=\tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha=\frac{1}{\tan\alpha}=\frac{3}{4}$

Correct Answer: C

45. Given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0<b<a)$, V_1 and V_2 are its left vertex and right vertex respectively. F_1 and F_2 are its left focus and right focus respectively. Let $|V_1F_1|$, $|F_1F_2|$, and $|F_1V_2|$ denote the distances between these points. They form a geometric sequence. Find the eccentricity of the ellipse.

A. $1/\sqrt{5}$ B. $1/4$ C. $1/2$ D. $3/5$ E. $\sqrt{5}$

Let $(\pm c, 0)$ be the coordinates of the foci of the ellipse. We have $|V_1F_1|=a-c$, $|F_1F_2|=2c$, and $|F_1V_2|=a+c$. Since they form a geometric sequence, $(a-c)(a+c)=(2c)^2$. $a^2-c^2=4c^2$, $5c^2=a^2$, and $e=\frac{c}{a}=\frac{1}{\sqrt{5}}$.

Correct Answer: A