1. Consider the piecewise function $f(x)=\begin{cases} 8x^3 & x<0 \\ -\sin x & 0\le x \end{cases}$. Find $f\left(f\left(\frac{\pi}{3}\right)\right)$.

$f\left(f\left(\frac{\pi}{3}\right)\right)=f\left(-\sin\frac{\pi}{3}\right)=f\left(-\frac{\sqrt{3}}{2}\right)=8\left(-\frac{\sqrt{3}}{2}\right)^3=-3\sqrt{3}$

Correct Answer: $-3\sqrt{3}$

2. A cubic foot is what fraction of a cubic yard?

A foot is equal to $\frac{1}{3}$ of a yard. A cubic foot is equal to $\left(\frac{1}{3}\right)^3=\frac{1}{27}$ of a cubic yard.

Correct Answer: $\frac{1}{27}$

3. If $3^x-3^{x-1}=162$, then $x=?$

$3^x-\frac{3^x}{3}=162$, $\frac{2}{3}\cdot 3^x=162$, $3^x=243=3^5$, and $x=5$

Correct Answer: 5

4. What is the probability of rolling a 7 at least once in two rolls of a pair of fair, six-sided dice?

The probability of not rolling a 7 is $\frac{5}{6}$. Therefore, the probability of not rolling a 7 in two rolls is $\frac{5}{6}\cdot\frac{5}{6}=\frac{25}{36}$. Thus the probability of rolling a 7 at least once in two rolls is $1-\frac{25}{36}=\frac{11}{36}$.

Correct Answer: $\frac{11}{36}$

5. Mulletonium doubles in volume every minute. At 09:00 a small amount is placed in a test container, and at 10:00 the container has just reached the full mark. At what time was the container one quarter full?

Since the container is full at 10:00, it is $1/2$ full at 09:59 and $1/4$ full at 09:58.

Correct Answer: 09:58

6. Given $f(x)=\dfrac{x^3}{x^2+1}$, what is x if $f^{-1}(x)=2$?

$$x=f\left(f^{-1}(x)\right)=f(2)=\frac{2^3}{2^2+1}=\frac{8}{5}$$

Correct Answer: $\dfrac{8}{5}$

7. If $2\tan\left(\cos^{-1}\left(\dfrac{3}{5}\right)\right)=x$, then $x=?$

Let $\theta=\cos^{-1}\left(\dfrac{3}{5}\right)$, $\cos\theta=\dfrac{3}{5}$, $\tan\theta=\dfrac{4}{3}$, and $2\tan\left(\cos^{-1}\left(\dfrac{3}{5}\right)\right)=2\tan\theta=2\cdot\dfrac{4}{3}=\dfrac{8}{3}$

Correct Answer: $\dfrac{8}{3}$

8. A father budgeted \$24 to distribute equally among his children for spending money at a pool party. When two young cousins joined the party and shared in equal distribution of this money, each child in the party received \$1 less than originally planned. How many children in total were at the party?

Let x be the number of children in total. Then we have $\dfrac{24}{x}=\dfrac{24}{x-2}-1$,

$24(x-2)=24x-x(x-2)$, $x^2-2x-48=0$, $(x+6)(x-8)=0$, and $x=8$

Correct Answer: 8

9. Determine the numerical value of the following and write the result in simplest, reduced form:

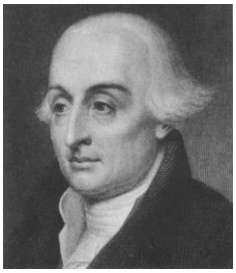
$$\frac{\tan(90^\circ - x) + \tan(x - 30^\circ)}{1 - \tan(90^\circ - x)\tan(x - 30^\circ)}$$

$$\frac{\tan(90^\circ - x) + \tan(x - 30^\circ)}{1 - \tan(90^\circ - x)\tan(x - 30^\circ)}$$

$$= \tan(90^\circ - x + x - 30^\circ) = \tan 60^\circ = \sqrt{3}$$

Correct Answer: $\sqrt{3}$

10. Who am I?



1736-1813

Hint: "Home, Home, on the Lag _ _ _ _ _"

Correct Answer: Joseph Lagrange

11. Angle x is nine times angle y; while the complement of angle y is nine times the complement of angle x. Find angle y to the nearest degree.

Solve the system $\begin{cases} 9y = x & (1) \\ 90^\circ - y = 9(90^\circ - x) & (2) \end{cases}$ by eliminating x.

Multiplying equation (1) by 9, then add the result to equation (2). We have $90^\circ + 80y = 810^\circ$.

Thus, $y = 9^\circ$.

Correct Answer: 9°

12. $\cos\left(\frac{\pi}{2}-x\right)\csc(x)=?$

$\cos\left(\frac{\pi}{2}-x\right)\csc x=\sin x\cdot\frac{1}{\sin x}=1$

Correct Answer: 1

13. Arrange the following three numbers from smallest to largest.

$a=\log_2 3,\quad b=\log_{1/2}\frac{1}{5},\quad c=\log_3 2$

$a=\log_2 3=\frac{\ln 3}{\ln 2}$, $b=\log_{1/2}\frac{1}{5}=\frac{\ln\frac{1}{5}}{\ln\frac{1}{2}}=\frac{-\ln 5}{-\ln 2}=\frac{\ln 5}{\ln 2}$, and $c=\log_3 2=\frac{\ln 2}{\ln 3}$.

We have $c<a<b$.

Correct Answer: $c<a<b$ or $\log_3 2<\log_2 3<\log_{1/2}\frac{1}{5}$

14. What is the area of a square whose diagonal is one unit longer than the length of its side?

Let *x* be the length of the side of the square. Its diagonal is $x\sqrt{2}$. We have $x\sqrt{2}=x+1$.
$2x^2=x^2+2x+1$, $x^2-2x=1$, $(x-1)^2=2$, $x=1+\sqrt{2}$, and $x^2=3+2\sqrt{2}$.

Correct Answer: $3+2\sqrt{2}$ or $\frac{1}{3-2\sqrt{2}}$

15. If $x=\cos^2 10^\circ+\cos^2 20^\circ+\cdots+\cos^2 70^\circ+\cos^2 80^\circ$, express the rational number *x* in the simplest, reduced form.

$\cos^2 10^\circ+\cos^2 20^\circ+\cdots+\cos^2 70^\circ+\cos^2 80^\circ$
$=\cos^2 10^\circ+\cos^2 20^\circ+\cdots+\cos^2(90^\circ-20^\circ)+\cos^2(90^\circ-10^\circ)$
$=\cos^2 10^\circ+\cos^2 20^\circ+\cdots+\sin^2 20^\circ+\sin^2 10^\circ=1+1+1+1=4$

Correct Answer: 4

16. Determine the value of the following in simplest form: $a^{\log_{a^2} 5}$

$$a^{\log_{a^2} 5} = \left(a^{2\cdot\frac{1}{2}} \right)^{\log_{a^2} 5} = \left(a^2\right)^{\frac{1}{2}\log_{a^2} 5} = \left(a^2\right)^{\log_{a^2} 5^{1/2}} = 5^{1/2} = \sqrt{5}$$

Correct Answer: $\sqrt{5}$ or $5^{1/2}$

17. Find the fraction that is equal to the repeating decimal $0.201420142014\cdots$.

$$0.201420142014\cdots = \frac{2014}{10000} + \frac{2014}{10000^2} + \frac{2014}{10000^3} + \cdots$$
$$= \frac{2014}{10000} \cdot \frac{1}{1-\frac{1}{10000}} = \frac{2014}{9999}$$

Correct Answer: $\frac{2014}{9999}$

18. A and B are independent events such that $P(A \mid B) = \frac{1}{2}$ and $P(B \mid A) = \frac{1}{4}$. What is $P(A \cap B)$?

$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$ and $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{4}$.

Then $\frac{P(A \cap B)}{P(B)} \cdot \frac{P(A \cap B)}{P(A)} = \frac{(P(A \cap B))^2}{P(B)P(A)} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ Since A and B are independent events, we have $P(B)P(A) = P(A \cap B)$. Therefore, $P(A \cap B) = \frac{1}{8}$

Correct Answer: $\frac{1}{8}$

19. What is the value of $\sqrt{10+4\sqrt{6}}-\sqrt{10-4\sqrt{6}}$?

$$\sqrt{10+4\sqrt{6}}-\sqrt{10-4\sqrt{6}}$$
$$=\sqrt{\left(\sqrt{6}\right)^2+2\cdot 2\sqrt{6}+2^2}-\sqrt{\left(\sqrt{6}\right)^2-2\cdot 2\sqrt{6}+2^2}$$
$$=\sqrt{\left(\sqrt{6}+2\right)^2}-\sqrt{\left(\sqrt{6}-2\right)^2}=4$$

Correct Answer: 4

20. Who am I?

1862 –1943

Hint: I proposed ten mathematical problems in 1900.

Correct Answer: David Hilbert